An Introduction to the Wavelet Analysis
of Time Series

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Overview

• wavelets are analysis tools mainly for
  – time series analysis (focus of this tutorial)
  – image analysis (will not cover)

• as a subject, wavelets are
  – relatively new (1983 to present)
  – synthesis of many new/old ideas
  – keyword in 10,558+ articles & books since 1989
    (2000+ in the last year alone)

• broadly speaking, have been two waves of wavelets
  – continuous wavelet transform (1983 and on)
  – discrete wavelet transform (1988 and on)
Game Plan

• introduce subject via CWT

• describe DWT and its main ‘products’
  – multiresolution analysis (additive decomposition)
  – analysis of variance (‘power’ decomposition)

• describe selected uses for DWT
  – wavelet variance (related to Allan variance)
  – decorrelation of fractionally differenced processes
    (closely related to power law processes)
  – signal extraction (denoising)
What is a Wavelet?

• wavelet is a ‘small wave’ (sinusoids are ‘big waves’)
• real-valued $\psi(t)$ is a wavelet if
  1. integral of $\psi(t)$ is zero: $\int_{-\infty}^{\infty} \psi(t) \, dt = 0$
  2. integral of $\psi^2(t)$ is unity: $\int_{-\infty}^{\infty} \psi^2(t) \, dt = 1$
     (called ‘unit energy’ property)
• wavelets so defined deserve their name because
  – #2 says we have, for every small $\epsilon > 0$,
    $$\int_{-T}^{T} \psi^2(t) \, dt < 1 - \epsilon,$$
    for some finite $T$ (might be quite large!)
  – length of $[-T, T]$ small compare to $[-\infty, \infty]$
  – #2 says $\psi(t)$ must be nonzero somewhere
  – #1 says $\psi(t)$ balances itself above/below 0
• Fig. 1: three wavelets
• Fig. 2: examples of complex-valued wavelets
**Basics of Wavelet Analysis: I**

- wavelets tell us about variations in local averages
- to quantify this description, let \( x(t) \) be a ‘signal’
  - real-valued function of \( t \)
  - will refer to \( t \) as time (but can be, e.g., depth)
- consider average value of \( x(t) \) over \([a, b]\):
  \[
  \frac{1}{b-a} \int_{a}^{b} x(u) \, du \equiv \alpha(a, b)
  \]
- reparameterize in terms of \( \lambda \) & \( t \)
  \[
  A(\lambda, t) \equiv \alpha(t - \frac{\lambda}{2}, t + \frac{\lambda}{2}) = \frac{1}{\lambda} \int_{t - \frac{\lambda}{2}}^{t + \frac{\lambda}{2}} x(u) \, du
  \]
  - \( \lambda \equiv b - a \) is called scale
  - \( t = (a + b)/2 \) is center time of interval
- \( A(\lambda, t) \) is average value of \( x(t) \) over scale \( \lambda \) at \( t \)
Basics of Wavelet Analysis: II

• average values of signals are of wide-spread interest
  – hourly rainfall rates
  – monthly mean sea surface temperatures
  – yearly average temperatures over central England
  – etc., etc., etc. (Rogers & Hammerstein, 1951)

• Fig. 3: fractional frequency deviates in clock 571
  – can regard as averages of form \([t - \frac{1}{2}, t + \frac{1}{2}]\)
  – \(t\) is measured in days (one measurement per day)
  – plot shows \(A(1, t)\) versus integer \(t\)
  – \(A(1, t) = 0\) ⇒ master clock & 571 agree perfectly
  – \(A(1, t) < 0\) ⇒ clock 571 is losing time
  – can easily correct if \(A(1, t)\) constant
  – quality of clock related to changes in \(A(1, t)\)
Basics of Wavelet Analysis: III

• can quantify changes in $A(1, t)$ via

$$D(1, t - \frac{1}{2}) \equiv A(1, t) - A(1, t - 1)$$

$$= \frac{t+\frac{1}{2}}{t-\frac{1}{2}} x(u) du - \frac{t-\frac{1}{2}}{t-\frac{3}{2}} x(u) du,$$

or, equivalently,

$$D(1, t) = A(1, t + \frac{1}{2}) - A(1, t - \frac{1}{2})$$

$$= \frac{t+1}{t} x(u) du - \frac{t}{t-1} x(u) du$$

• generalizing to scales other than unity yields

$$D(\lambda, t) \equiv A(\lambda, t + \frac{\lambda}{2}) - A(\lambda, t - \frac{\lambda}{2})$$

$$= \frac{1}{\lambda} \frac{t+\lambda}{t} x(u) du - \frac{1}{\lambda} \frac{t}{t-\lambda} x(u) du$$

• $D(\lambda, t)$ often of more interest than $A(\lambda, t)$

• can connect to Haar wavelet: write

$$D(\lambda, t) = \int_{-\infty}^{\infty} \tilde{\psi}_{\lambda,t}(u)x(u) du$$

with

$$\tilde{\psi}_{\lambda,t}(u) \equiv \begin{cases} 
-1/\lambda, & t - \lambda \leq u < t; \\
1/\lambda, & t \leq u < t + \lambda; \\
0, & \text{otherwise}.
\end{cases}$$
Basics of Wavelet Analysis: IV

• specialize to case $\lambda = 1$ and $t = 0$:

\[
\tilde{\psi}_{1,0}(u) \equiv \begin{cases} 
-1, & -1 \leq u < 0; \\
1, & 0 \leq u < 1; \\
0, & \text{otherwise.}
\end{cases}
\]

comparison to $\psi^H(u)$ yields $\tilde{\psi}_{1,0}(u) = \sqrt{2}\psi^H(u)$

• Haar wavelet mines out info on difference between unit scale averages at $t = 0$ via

\[
\int_{-\infty}^{\infty} \psi^H(u)x(u) \, du \equiv W^H(1, 0)
\]

• to mine out info at other $t$'s, just shift $\psi^H(u)$:

\[
\psi_{1,t}(u) \equiv \psi^H(u-t); \quad \text{i.e.,} \quad \psi_{1,t}(u) = \begin{cases} 
-\frac{1}{\sqrt{2}}, & t - 1 \leq u < t; \\
\frac{1}{\sqrt{2}}, & t \leq u < t + 1; \\
0, & \text{otherwise}
\end{cases}
\]

Fig. 4: top row of plots

• to mine out info about other $\lambda$'s, form

\[
\psi_{\lambda,t}(u) \equiv \frac{1}{\sqrt{\lambda}}\psi^H\left(\frac{u - t}{\lambda}\right) = \begin{cases} 
-\frac{1}{\sqrt{2\lambda}}, & t - \lambda \leq u < t; \\
\frac{1}{\sqrt{2\lambda}}, & t \leq u < t + \lambda; \\
0, & \text{otherwise.}
\end{cases}
\]

Fig. 4: bottom row of plots
Basics of Wavelet Analysis: V

- can check that $\psi_{\lambda,t}^H(u)$ is a wavelet for all $\lambda$ & $t$
- use $\psi_{\lambda,t}^H(u)$ to obtain
  \[
  W^H(\lambda, t) \equiv \int_{-\infty}^{\infty} \psi_{\lambda,t}^H(u)x(u) \, du \propto D(\lambda, t)
  \]
  left-hand side is Haar CWT
- can do the same with other wavelets:
  \[
  W(\lambda, t) \equiv \int_{-\infty}^{\infty} \psi_{\lambda,t}(u)x(u) \, du, \text{ where } \psi_{\lambda,t}(u) \equiv \frac{1}{\sqrt{\lambda}}\psi\left(\frac{u - t}{\lambda}\right)
  \]
  left-hand side is CWT based on $\psi(u)$
- interpretation for $\psi^{fdG}(u)$ and $\psi^{Mh}(u)$ (Fig. 1): differences of adjacent weighted averages
Basics of Wavelet Analysis: VI

- basic CWT result: if $\psi(u)$ satisfies admissibility condition, can recover $x(t)$ from its CWT:

$$x(t) = \frac{1}{C_\psi} \mathcal{F}^{-1} \left[ W(\lambda, t) \frac{1}{\sqrt{\lambda}} \psi \left( \frac{t-u}{\lambda} \right) du \right] \frac{d\lambda}{\lambda^2},$$

where $C_\psi$ is constant depending just on $\psi$

- conclusion: $W(\lambda, t)$ equivalent to $x(t)$

- can also show that

$$\int_{-\infty}^{\infty} x^2(t) \, dt = \frac{1}{C_\psi} \left[ \int_{-\infty}^{\infty} W^2(\lambda, t) \, dt \right] \frac{d\lambda}{\lambda^2}$$

  - LHS called energy in $x(t)$
  - RHS integrand is energy density over $\lambda$ & $t$

- Fig. 3: Mexican hat CWT of clock 571 data
Beyond the CWT: the DWT

• critique: have transformed signal into an image
• can often get by with subsamples of $W(\lambda, t)$
• leads to notion of discrete wavelet transform (DWT)
  – can regard as dyadic ‘slices’ through CWT
  – can further subsample slices at various $t$’s
• DWT has appeal in its own right
  – most time series are sampled as discrete values
    (can be tricky to implement CWT)
  – can formulate as orthonormal transform
    (facilitates statistical analysis)
  – approximately decorrelates certain time series
    (including power law processes)
  – standardization to dyadic scales often adequate
  – can be faster than the fast Fourier transform!
• will concentrate on DWT for remainder of tutorial
Overview of DWT

• let \( X = [X_0, X_1, \ldots, X_{N-1}]^T \) be observed time series (for convenience, assume \( N \) integer multiple of \( 2^{J_0} \))

• let \( \mathcal{W} \) be \( N \times N \) orthonormal DWT matrix

• \( W = \mathcal{W}X \) is vector of DWT coefficients

• orthonormality says \( X = \mathcal{W}^T W \), so \( X \Leftrightarrow W \)

• can partition \( W \) as follows:

\[
W = \begin{bmatrix}
W_1 \\
\vdots \\
W_{J_0} \\
V_{J_0}
\end{bmatrix}
\]

• \( W_j \) contains \( N_j = N/2^j \) wavelet coefficients
  – related to changes of averages at scale \( \tau_j = 2^{j-1} \) (\( \tau_j \) is \( j \)th ‘dyadic’ scale)
  – related to times spaced \( 2^j \) units apart

• \( V_{J_0} \) contains \( N_{J_0} = N/2^{J_0} \) scaling coefficients
  – related to averages at scale \( \lambda_{J_0} = 2^{J_0} \)
  – related to times spaced \( 2^{J_0} \) units apart
Example: Haar DWT

- Fig. 5: $W$ for Haar DWT with $N = 16$
  - first 8 rows yield $W_1 \propto changes$ on scale 1
  - next 4 rows yield $W_2 \propto changes$ on scale 2
  - next 2 rows yield $W_3 \propto changes$ on scale 4
  - next to last row yields $W_4 \propto change$ on scale 8
  - last row yields $V_4 \propto average$ on scale 16

- Fig. 6: Haar DWT coefficients for clock 571
DWT in Terms of Filters

- filter $X_0, X_1, \ldots, X_{N-1}$ to obtain
  \[2^{j/2} \tilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} h_{j,l} X_{t-l \mod N}, \quad t = 0, 1, \ldots, N - 1\]
  where $h_{j,l}$ is $j$th level wavelet filter
  - note: circular filtering

- subsample to obtain wavelet coefficients:
  \[W_{j,t} = 2^{j/2} \tilde{W}_{j,2^{j}(t+1)-1}, \quad t = 0, 1, \ldots, N_j - 1,\]
  where $W_{j,t}$ is $t$th element of $W_j$

- Figs. 7 & 8: Haar, D(4), C(6) & LA(8) wavelet filters

- $j$th wavelet filter is band-pass with pass-band $[\frac{1}{2^j+1}, \frac{1}{2^j}]$
- note: $j$th scale related to interval of frequencies

- similarly, scaling filters yield $V_{J0}$

- Figs. 9 & 10: Haar, D(4), C(6) & LA(8) scaling filters

- $J_0$th scaling filter is low-pass with pass-band $[0, \frac{1}{2^{J_0+1}}]$
Pyramid Algorithm: I

• can formulate DWT via ‘pyramid algorithm’
  – elegant iterative algorithm for computing DWT
  – implicitly defines $W$
  – computes $W = WX$ using $O(N)$ multiplications
    * ‘brute force’ method uses $O(N^2)$
    * FFT algorithm uses $O(N \log_2 N)$
• algorithm makes use of two basic filters
  – wavelet filter $h_l$ of unit scale $h_l \equiv h_{1,l}$
  – associated scaling filter $g_l$
The Wavelet Filter: I

• let $h_l, l = 0, \ldots, L - 1,$ be a real-valued filter
  - $L$ is filter width so $h_0 \neq 0 \& h_{L-1} \neq 0$
  - $L$ must be even
  - assume $h_l = 0$ for $l < 0 \& l \geq L$

• $h_l$ called a wavelet filter if it has these 3 properties
  1. summation to zero:
     $$\sum_{l=0}^{L-1} h_l = 0$$
  2. unit energy:
     $$\sum_{l=0}^{L-1} h_l^2 = 1$$
  3. orthogonality to even shifts:
     $$\sum_{l=0}^{L-1} h_l h_{l+2n} = \sum_{l=-\infty}^{\infty} h_l h_{l+2n} = 0$$
     for all nonzero integers $n$

• 2 & 3 together called orthonormality property
The Wavelet Filter: II

• transfer & squared gain functions for \( h_l \):

\[
H(f) \equiv \sum_{l=0}^{L-1} h_l e^{-i2\pi fl} \quad \text{&} \quad \mathcal{H}(f) \equiv |H(f)|^2
\]

• can argue that orthonormality property equivalent to

\[
\mathcal{H}(f) + \mathcal{H}(f + \frac{1}{2}) = 2 \quad \text{for all } f
\]

• Fig. 11: \( \mathcal{H}(f) \) for Daubechies wavelet filters
  – \( L = 2 \) case is Haar wavelet filter
  – filter cascade with averaging & differencing filters
  – high-pass filter with pass-band \([\frac{1}{4}, \frac{1}{2}]\)
  – can regard as half-band filter
The Scaling Filter: I

- scaling filter: \( g_l \equiv (-1)^{l+1} h_{L-1-l} \)
  - reverse \( h_l \) & flip sign of every other coefficient
  - e.g.: \( h_0 = \frac{1}{\sqrt{2}} \) & \( h_1 = -\frac{1}{\sqrt{2}} \) \( \Rightarrow \)
    \( g_0 = g_1 = \frac{1}{\sqrt{2}} \)
  - \( g_l \) is ‘quadrature mirror’ filter for \( h_l \)

- properties of \( h_l \) imply \( g_l \) has these properties:
  1. summation to \( \pm \sqrt{2} \), so will assume
     \[ \sum_{l=0}^{L-1} g_l = \sqrt{2} \]
  2. unit energy:
     \[ \sum_{l=0}^{L-1} g_l^2 = 1 \]
  3. orthogonality to even shifts:
     \[ \sum_{l=0}^{L-1} g_l g_{l+2n} = \sum_{l=-\infty}^{\infty} g_l g_{l+2n} = 0 \]
     for all nonzero integers \( n \)
  4. orthogonality to wavelet filter at even shifts:
     \[ \sum_{l=0}^{L-1} g_l h_{l+2n} = \sum_{l=-\infty}^{\infty} g_l h_{l+2n} = 0 \]
     for all integers \( n \)
The Scaling Filter: II

- transfer & squared gain functions for $g_l$:

$$ G(f) \equiv \sum_{l=0}^{L-1} g_l e^{-i2\pi fl} \quad \& \quad \mathcal{G}(f) \equiv |G(f)|^2 $$

- can argue that $\mathcal{G}(f) = \mathcal{H}(f - \frac{1}{2})$
  - have $\mathcal{G}(0) = \mathcal{H}(-\frac{1}{2}) = \mathcal{H}(\frac{1}{2})$ & $\mathcal{G}(\frac{1}{2}) = \mathcal{H}(0)$
  - since $h_l$ is high-pass, $g_l$ must be low-pass
  - low-pass filter with pass-band $[0, \frac{1}{4}]$
  - can also regard as half-band filter

- orthonormality property equivalent to

$$ \mathcal{G}(f) + \mathcal{G}(f + \frac{1}{2}) = 2 \quad \& \quad \mathcal{H}(f) + \mathcal{G}(f) = 2 \quad \text{for all } f $$
Pyramid Algorithm: II

• define $V_0 \equiv X$ and set $j = 1$

• input to $j$th stage of pyramid algorithm is $V_{j-1}$
  – $V_{j-1}$ is full-band
  – related to frequencies $[0, \frac{1}{2j}]$ in $X$

• filter with half-band filters and downsample:

$$W_{j,t} \equiv \sum_{l=0}^{L-1} h_l V_{j-1,2t+1-l \mod N_{j-1}}$$
$$V_{j,t} \equiv \sum_{l=0}^{L-1} g_l V_{j-1,2t+1-l \mod N_{j-1}},$$

$t = 0, \ldots, N_j - 1$

• place these in vectors $W_j$ & $V_j$
  – $W_j$ are wavelet coefficients for scale $\tau_j = 2^{j-1}$
  – $V_j$ are scaling coefficients for scale $\lambda_j = 2^j$

• increment $j$ and repeat above until $j = J_0$

• yields DWT coefficients $W_1, \ldots, W_{J_0}, V_{J_0}$
Pyramid Algorithm: III

- can formulate inverse pyramid algorithm (recovers $V_{j-1}$ from $W_j$ and $V_j$)
- algorithm implicitly defines transform matrix $W$
- partition $W$ commensurate with $W_j$:

$$W = \begin{bmatrix}
W_1 \\
W_2 \\
\vdots \\
W_{J_0} \\
V_{J_0}
\end{bmatrix} \quad \text{parallels} \quad W = \begin{bmatrix}
W_1 \\
W_2 \\
\vdots \\
W_{J_0} \\
V_{J_0}
\end{bmatrix}$$

- rows of $W_j$ use $j$th level filter $h_{j,l}$ with DFT

$$H(2^{j-1}f) \prod_{l=0}^{j-2} G(2^l f)$$

($h_{j,l}$ has $L_j = (2^j - 1)(L - 1) + 1$ nonzero elements)

- $W_j$ is $N_j \times N$ matrix such that

$$W_j = W_jX \quad \text{and} \quad W_j W_j^T = I_{N_j}$$
Two Consequences of Orthonormality

• multiresolution analysis (MRA)

$$\mathbf{X} = \mathbf{W}^T \mathbf{W} = \sum_{j=1}^{J_0} \mathbf{W}_j^T \mathbf{W}_j + \mathbf{V}_{J_0} \mathbf{V}_{J_0} \equiv \sum_{j=1}^{J_0} \mathbf{D}_j + \mathbf{S}_{J_0}$$

- scale-based additive decomposition
- $\mathbf{D}_j$'s & $\mathbf{S}_{J_0}$ called details & smooth

• analysis of variance

- consider ‘energy’ in time series:

$$\|\mathbf{X}\|^2 = \mathbf{X}^T \mathbf{X} = \sum_{t=0}^{N-1} X_t^2$$

- energy preserved in DWT coefficients:

$$\|\mathbf{W}\|^2 = \|\mathbf{W} \mathbf{X}\|^2 = \mathbf{X}^T \mathbf{W}^T \mathbf{W} \mathbf{X} = \mathbf{X}^T \mathbf{X} = \|\mathbf{X}\|^2$$

- since $\mathbf{W}_1, \ldots, \mathbf{W}_{J_0}, \mathbf{V}_{J_0}$ partitions $\mathbf{W}$, have

$$\|\mathbf{W}\|^2 = \sum_{j=1}^{J_0} \|\mathbf{W}_j\|^2 + \|\mathbf{V}_{J_0}\|^2,$$

leading to analysis of sample variance:

$$\hat{\sigma}^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \bar{X})^2 = \frac{1}{N} \sum_{j=1}^{J_0} \|\mathbf{W}_j\|^2 + \left( \frac{1}{N} \|\mathbf{V}_{J_0}\|^2 - \bar{X}^2 \right)$$

- scale-based decomposition (cf. frequency-based)
Variation: Maximal Overlap DWT

- can eliminate downsampling and use

\[ W_{j,t} \equiv \frac{1}{2^{j/2}} \sum_{l=0}^{L_j-1} h_{j,l} x_{t-l \mod N}, \quad t = 0, 1, \ldots, N - 1 \]

to define MODWT coefficients \( W_j \) (& also \( V_j \))
- unlike DWT, MODWT is not orthonormal
  (in fact MODWT is highly redundant)
- like DWT, can do MRA & analysis of variance:

\[
\|X\|^2 = \sum_{j=1}^{J_0} \|W_j\|^2 + \|V_{J_0}\|^2
\]
- unlike DWT, MODWT works for all samples sizes \( N \)
  (i.e., power of 2 assumption is not required)
  - if \( N \) is power of 2, can compute MODWT
    using \( O(N \log_2 N) \) operations
    (i.e., same as FFT algorithm)
  - contrast to DWT, which uses \( O(N) \) operations

- Fig. 12: Haar MODWT coefficients for clock 571
  (cf. Fig. 6 with DWT coefficients)
Definition of Wavelet Variance

• let $X_t, t = \ldots, -1, 0, 1, \ldots$, be a stochastic process

• run $X_t$ through $j$th level wavelet filter:

$$W_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t = \ldots, -1, 0, 1, \ldots,$$

which should be contrasted with

$$W_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \mod N}, \quad t = 0, 1, \ldots, N - 1$$

• definition of time dependent wavelet variance
  (also called wavelet spectrum):

$$\nu_{X,t}^2(\tau_j) \equiv \text{var}\{W_{j,t}\},$$

assuming $\text{var}\{W_{j,t}\}$ exists and is finite

• $\nu_{X,t}^2(\tau_j)$ depends on $\tau_j$ and $t$

• will consider time independent wavelet variance:

$$\nu_X^2(\tau_j) \equiv \text{var}\{W_{j,t}\}$$

(can be easily adapted to time varying situation)
Rationale for Wavelet Variance

• decomposes variance on scale by scale basis
• useful substitute/complement for spectrum
• useful substitute for process/sample variance
Variance Decomposition

• suppose $X_t$ has power spectrum $S_X(f)$:

$$\frac{1}{2} \int_{-1/2}^{1/2} S_X(f) \, df = \text{var} \{ X_t \};$$

i.e., decomposes $\text{var} \{ X_t \}$ across frequencies $f$

- involves uncountably infinite number of $f$’s
- $S_X(f) \Delta f \approx$ contribution to $\text{var} \{ X_t \}$ due to $f$’s in interval of length $\Delta f$ centered at $f$

• wavelet variance analog to fundamental result:

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var} \{ X_t \}$$

i.e., decomposes $\text{var} \{ X_t \}$ across scales $\tau_j$

- recall DWT/MODWT and sample variance
- involves countably infinite number of $\tau_j$’s
- $\nu_X^2(\tau_j)$ contribution to $\text{var} \{ X_t \}$ due to scale $\tau_j$
- $\nu_X(\tau_j)$ has same units as $X_t$ (easier to interpret)
Spectrum Substitute/Complement

• because $\tilde{h}_{j,l} \approx$ bandpass over $[1/2^{j+1}, 1/2^j]$,  
\[ \nu^2_X(\tau_j) \approx 2^{1/2^j} \int_{1/2^{j+1}}^{1/2^j} S_X(f) \, df \]

• if $S_X(f)$ ‘featureless’, info in $\nu^2_X(\tau_j) \Leftrightarrow$ info in $S_X(f)$

• $\nu^2_X(\tau_j)$ more succinct: only 1 value per octave band

• example: $S_X(f) \propto |f|^\alpha$, i.e., power law process
  
  – can deduce $\alpha$ from slope of log $S_X(f)$ vs. log $f$
  – implies $\nu^2_X(\tau_j) \propto \tau_j^{-\alpha-1}$ approximately
  
  – can deduce $\alpha$ from slope of log $\nu^2_X(\tau_j)$ vs. log $\tau_j$
  
  – no loss of ‘info’ using $\nu^2_X(\tau_j)$ rather than $S_X(f)$

• with Haar wavelet, obtain pilot spectrum estimate proposed in Blackman & Tukey (1958)
Substitute for Variance: I

• can be difficult to estimate process variance
• $\nu^2_X(\tau_j)$ useful substitute: easy to estimate & finite
• let $\mu = E\{X_t\}$ be known, $\sigma^2 = \text{var}\{X_t\}$ unknown
• can estimate $\sigma^2$ using
  \[ \tilde{\sigma}^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \mu)^2 \]
  • estimator above is unbiased: $E\{\tilde{\sigma}^2\} = \sigma^2$
• if $\mu$ is unknown, can estimate $\sigma^2$ using
  \[ \hat{\sigma}^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \bar{X})^2 \]
  • there is some (non-pathological!) $X_t$ such that
  \[ \frac{E\{\hat{\sigma}^2\}}{\sigma^2} < \epsilon \]
  for any given $\epsilon > 0$ & $N \geq 1$
• $\hat{\sigma}^2$ can badly underestimate $\sigma^2$!
• example: power law process with $-1 < \alpha < 0$
Substitute for Variance: II

- Q: why is wavelet variance useful when $\sigma^2$ is not?
- replaces ‘global’ variability with variability over scales
- if $X_t$ stationary with mean $\mu$, then

$$E\{W_{j,t}\} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} E\{X_{t-l}\} = \mu \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} = 0$$

because $\sum_l \tilde{h}_{j,l} = 0$

- $E\{W_{j,t}\}$ known, so can get unbiased estimator of

$$\text{var} \{W_{j,t}\} = \nu_X^2(\tau_j)$$

- certain nonstationary $X_t$ have well-defined $\nu_X^2(\tau_j)$
- example: power law processes with $\alpha \leq -1$
  (example of process with stationary increments)
Estimation of Wavelet Variance: I

• can base estimator on MODWT of $X_0, X_1, \ldots, X_{N-1}$:

$$W_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \mod N}, \quad t = 0, 1, \ldots, N - 1$$

(DWT-based estimator possible, but less efficient)

• recall that

$$\tilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t = 0, \pm 1, \pm 2, \ldots$$

so $W_{j,t} = \tilde{W}_{j,t}$ if mod not needed: $L_j - 1 \leq t < N$

• if $N - L_j \geq 0$, unbiased estimator of $\nu_X^2(\tau_j)$ is

$$\hat{\nu}_X^2(\tau_j) \equiv \frac{1}{N - L_j + 1} \sum_{t=L_j-1}^{N-1} W_{j,t}^2 = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \tilde{W}_{j,t}^2,$$

where $M_j \equiv N - L_j + 1$

• can also construct biased estimator of $\nu_X^2(\tau_j)$:

$$\tilde{\nu}_X^2(\tau_j) \equiv \frac{1}{N} \sum_{t=0}^{N-1} W_{j,t}^2 = \frac{1}{N} \left( \sum_{t=0}^{L_j-2} W_{j,t}^2 + \sum_{t=L_j-1}^{N-1} \tilde{W}_{j,t}^2 \right)$$

1st sum in parentheses influenced by circularity
Estimation of Wavelet Variance: II

- biased estimator unbiased if $\{X_t\}$ white noise
- biased estimator offers exact analysis of $\hat{\sigma}^2$; unbiased estimator need not
- biased estimator can have better mean square error (Greenhall et al., 1999; need to ‘reflect’ $X_t$)
Statistical Properties of $\hat{\nu}_X^2(\tau_j)$

- suppose $\{W_{j,t}\}$ Gaussian, mean 0 & spectrum $S_j(f)$
- suppose square integrability condition holds:
  $$A_j \equiv \frac{1}{2} \int_{-1/2}^{1/2} S_j^2(f) \, df < \infty \ & \ S_j(f) > 0$$
  (holds for power law processes if $L$ large enough)
- can show $\hat{\nu}_X^2(\tau_j)$ asymptotically normal with
  mean $\nu_X^2(\tau_j)$ & large sample variance $2A_j/M_j$
- can estimate $A_j$ and use with $\hat{\nu}_X^2(\tau_j)$
  to construct confidence interval for $\nu_X^2(\tau_j)$
- example
  - Fig. 13: clock errors $X_t \equiv X_t^{(0)}$ along with
    differences $X_t^{(i)} \equiv X_t^{(i-1)} - X_{t-1}^{(i-1)}$ for $i = 1, 2$
  - Fig. 14: $\hat{\nu}_X^2(\tau_j)$ for clock errors
  - Fig. 15: $\hat{\nu}_Y^2(\tau_j)$ for $Y_t \propto X_t^{(1)}$
  - Haar $\hat{\nu}_Y^2(\tau_j)$ related to Allan variance $\sigma_Y^2(2, \tau_j)$:
    $$\nu_Y^2(\tau_j) = \frac{1}{2} \sigma_Y^2(2, \tau_j)$$
Decorrelation of FD Processes

- \( X_t \) ‘fractionally differenced’ if its spectrum is

\[
S_X(f) = \frac{\sigma^2}{|2\sin(\pi f)|^{2\delta}},
\]

where \( \sigma^2 > 0 \) and \(-\frac{1}{2} < \delta < \frac{1}{2}\)

- note: for small \( f \), have \( S_X(f) \approx C/|f|^{2\delta} \);
  i.e., power law with \( \alpha = -2\delta \)

- if \( \delta = 0 \), FD process is white noise

- if \( 0 < \delta < \frac{1}{2} \), FD stationary with ‘long memory’

- can extend definition to \( \delta \geq \frac{1}{2} \)
  - nonstationary \( 1/f \) type process
  - also called ARFIMA(0,\( \delta \),0) process

- Fig. 16: DWT of simulated FD process, \( \delta = 0.4 \)
  (sample autocorrelation sequences (ACSs) on right)
DWT as Whitening Transform

• sample ACSs suggest $W_j \approx$ uncorrelated

• since FD process is stationary, so are $W_j$
  (ignoring terms influenced by circularity)

• Fig. 17: spectra for $W_j$, $j = 1, 2, 3, 4$

• $W_j \& W_{j'}$, $j = j'$, approximately uncorrelated
  (approximation improves as $L$ increases)

• DWT thus acts as a whitening transform

• lots of uses for whitening property, including:
  
  1. testing for variance changes
  2. bootstrapping time series statistics
  3. estimating $\delta$ for stationary/nonstationary
     fractional difference processes with trend
Estimation for FD Processes: I

• extension of work by Wornell; McCoy & Walden

• problem: estimate $\delta$ from time series $U_t$ such that

$$U_t = T_t + X_t$$

where

$- T_t \equiv \sum_{j=0}^{r} a_j t^j$ is polynomial trend

$- X_t$ is FD process, but can have $\delta \geq \frac{1}{2}$

• DWT wavelet filter of width $L$ has embedded differencing operation of order $L/2$

• if $\frac{L}{2} \geq r + 1$, reduces polynomial trend to 0

• can partition DWT coefficients as

$$W = W_s + W_b + W_w$$

where

$- W_s$ has scaling coefficients and 0s elsewhere

$- W_s$ has boundary-dependent wavelet coefficients

$- W_w$ has boundary-independent wavelet coefficients
Estimation for FD Processes: II

- since $U = W^T W$, can write
  
  $$U = W^T (W_s + W_b) + W^T W_w \equiv \hat{T} + \hat{X}$$

- Fig. 18: example with fractional frequency deviates
  
  - can use values in $W_w$ to form likelihood:

  $$L(\delta, \sigma_{\epsilon}^2) \equiv J_0 \sum_{j=1}^{N_j'} \frac{1}{2\pi\sigma_j} e^{-W_{j,t+L_j'-1}/(2\sigma_j^2)}$$

  where

  $$\sigma_j^2 \equiv \frac{1}{2\pi \sigma_j} \frac{\sigma_{\epsilon}^2}{H_j(f)} \frac{1}{|2\sin(\pi f)|^2|2\delta|} df;$$

  and $H_j(f)$ is squared gain for $h_{j,l}$

  - leads to maximum likelihood estimator $\hat{\delta}$ for $\delta$
  - works well in Monte Carlo simulations
  - get $\hat{\delta} \approx 0.39 \pm 0.03$ for fractional frequency deviates
DWT-based Signal Extraction: I

• DWT analysis of $X$ yields $W = \mathcal{W}X$

• DWT synthesis $X = \mathcal{W}^T W$ yields
  
  – multiresolution analysis (MRA)
  
  – estimator of ‘signal’ $D$ hidden in $X$:
    
    * modify $W$ to get $W'$
    
    * use $W'$ to form signal estimate:
      
      $$\bar{D} \equiv \mathcal{W}^T W'$$

• key ideas behind wavelet-based signal estimation
  
  – DWT can isolate signals in small number of $W_n$’s
  
  – can ‘threshold’ or ‘shrink’ $W_n$’s

• key ideas lead to ‘waveshrink’
  
  (Donoho and Johnstone, 1995)
DWT-based Signal Extraction: II

• thresholding schemes involve

1. computing \( W \equiv \mathcal{W}X \)

2. defining \( W^{(t)} \) as vector with \( n \)th element

\[
W_n^{(t)} = \begin{cases} 
0, & \text{if } |W_n| \leq \delta; \\
\text{some nonzero value}, & \text{otherwise},
\end{cases}
\]

where nonzero values are yet to be defined

3. estimating \( D \) via \( \bar{D}^{(t)} \equiv \mathcal{W}^T W^{(t)} \)

• simplest scheme is ‘hard thresholding:’

\[
W_n^{(ht)} = \begin{cases} 
0, & \text{if } |W_n| \leq \delta; \\
W_n, & \text{otherwise}.
\end{cases}
\]

Fig. 19: solid line (‘kill/keep’ strategy)

• alternative scheme is ‘soft thresholding:’

\[
W_n^{(st)} = \text{sign} \{W_n\} (|W_n| - \delta)_+,
\]

where

\[
\text{sign} \{W_n\} \equiv \begin{cases} 
+1, & \text{if } W_n > 0; \\
0, & \text{if } W_n = 0; \\
-1, & \text{if } W_n < 0.
\end{cases}
\]

and \( (x)_+ \equiv \begin{cases} 
x, & \text{if } x \geq 0; \\
0, & \text{if } x < 0.
\end{cases} \)

Fig. 19: dashed line
DWT-based Signal Extraction: III

• third scheme is ‘mid thresholding:

\[ W_n^{(mt)} = \text{sign} \{ W_n \} (|W_n| - \delta)_{++}, \]

where

\[ (|W_n| - \delta)_{++} \equiv \begin{cases} 2(|W_n| - \delta)_{+}, & \text{if } |W_n| < 2\delta; \\ |W_n|, & \text{otherwise} \end{cases} \]

Fig. 19: dotted line

• Q: how should \( \delta \) be set?

• A: universal’ threshold (Donoho & Johnstone, 1995)
(lots of other answers have been proposed)

– specialize to model \( \mathbf{X} = \mathbf{D} + \mathbf{e} \),

where \( \mathbf{e} \) is Gaussian white noise with variance \( \sigma^2_e \)

– ‘universal’ threshold: \( \delta_U \equiv \sqrt{2\sigma^2_e \log(N)} \)

– rationale for \( \delta_U \):

* suppose \( \mathbf{D} = 0 \) & hence \( \mathbf{W} \) is white noise also
* as \( N \to \infty \), have

\[ \mathbf{P} \max_n |W_n| \leq \delta_U \to 1 \]

so all \( \mathbf{W}^{(ht)} = 0 \) with high probability
* will estimate correct \( \mathbf{D} \) with high probability
DWT-based Signal Extraction: IV

• can estimate $\sigma_e^2$ using median absolute deviation (MAD):

$$\hat{\sigma}_{(\text{MAD})} \equiv \frac{\text{median} \{|W_{1,0}|, |W_{1,1}|, \ldots, |W_{1,N-1}|\}}{0.6745},$$

where $W_{1,t}$'s are elements of $W_1$.

• Fig. 20: application to NMR series

• has potential application in dejamming GPS signals
  (with roles of ‘signal’ and ‘noise’ swapped!)
Web Material and Books

- Wavelet Digest
  
  http://www.wavelet.org/

- MathSoft’s wavelet resource page
  
  http://www.mathsoft.com/wavelets.html

- books
  
  
  
  
  
  http://www.staff.washington.edu/dbp/wmtsa.html

Software

• Matlab
  – WAVELAB (free):
    http://www-stat.stanford.edu/~wavelab
  – WAVEBOX (commercial):
    http://www.toolsmiths.com/

• Mathcad Wavelets Extension Pack (commercial):
  http://www.mathsoft.com/mathcad/ebooks/wavelets.asp

• S-Plus software
  – WAVETHRESH (free):
    http://lib.stat.cmu.edu/S/wavethresh
  – S+WAVELETS (commercial):
    http://www.mathsoft.com/splsprod/wavelets.html