Determination of Microwave Transducer and Delay-Line Properties with a Modified Nodal Shift Method

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Abstract—A mathematical analysis of the input impedance to a single-ended delay line is given and the developed formulas are applied to the experimental data to yield: sound velocity and round-trip time in the delay medium, electromechanical coupling coefficient, series resistance, transducer capacity, and the acoustical loss and mismatch which is caused by the electrode structure. A measuring technique similar to the nodal shift method has been introduced.

I. INTRODUCTION

FOR THE determination of the performance of microwave acoustic transducers and delay lines the standard approach is either by a pulse-echo technique, where the relative attenuation between delayed signals is evaluated,1 or CW methods such as input impedance or admittance measurements as a function of frequency where the results are customarily evaluated in terms of equivalent circuits.2 The method described below is of the CW type and can best be characterized as a modified nodal shift method, where the results are obtained from the input reflection coefficient as a function of frequency.

The experimentally observed loci for the input impedance of a single-ended delay line as a function of frequency appear in the form of circles in the impedance plane. Radii and center positions of these circles are analyzed in terms of series resistance, materials loss, electromechanical coupling coefficient, and impedance mismatch between transducer and delay medium. The analysis shows that the slight deviation from the circular form can be interpreted in terms of losses in the transducer material. In the development described below it becomes possible to separate parameters which to our knowledge cannot be determined independently by other methods.

II. THEORY

A. Input Impedance of a Single-Ended Delay Line

Although the following analysis can be carried out on the basis of a more general and more sophisticated model and for different configurations, it will be restricted here to a simplified case of a single-ended delay line. One frequently encounters this basic configuration in the development stage of microwave acoustic transducers where it is important to obtain accurately their performance characteristics.

The model on which the following analysis is based is shown in Fig. 1. A shear wave transducer (piezoelectric medium 1) sandwiched between two thin metal electrodes at \( x = 0 \) and \( x = x_1 \), is attached to a delay line (medium 2) of length \( x_2 - x_1 \). The electrical effect of the electrode structure is represented by a series impedance \( Z_0 \). The propagation constants for shear waves in the lossy media 1 and 2 are described by the complex quantities \( k_1 \) and \( k_2 \), respectively, while diffraction losses are not included in this analysis. The input impedance \( Z \) of the single ended delay line, as shown in Fig. 1, can be calculated readily either by solving the boundary condition problem or by the use of an equivalent circuit. The result of this calculation is

\[
Z = Z_0 + \frac{1}{j \omega C_0} \left( \frac{2(1 - \cos \theta) + j \xi \sin \theta + \lambda [2(1 - \cos \theta) - j \xi \sin \theta]}{\xi \cos \theta - j \sin \theta - \lambda [\xi \cos \theta + j \sin \theta]} \right)
\]

where the following abbreviations have been used:

\( Z_0 \) = series impedance of electrode structure (see Fig. 1)

\( C_0 \) = capacity of transducer far below resonance \((f \ll f_0)\)

\( f_0 \) = resonance frequency

\( k_1 \), \( k_2 \) = effective electromechanical coupling coefficient for shear waves

\( \lambda = \exp (2j k_2 x_2) \).

If \( k_1 \) and \( k_2 \) are written in the form

\[
k_i = k_{0i}(1 + j \delta_i) \quad \text{with} \quad k_{0i} = \text{Re} (k_i) \quad (i = 1, 2)
\]

and if the assumption is made that

\[
1 \gg \delta_1 \gg \delta_2 \quad \text{(or} \delta_1 - \delta_2 = \delta \approx \delta_1),
\]

then

\[
\frac{Z_2}{Z_1} = \frac{Z_2}{Z_1} = \frac{\frac{1}{1 + K^2}}{\frac{1}{\theta}}
\]

where

\[
K = \frac{1}{\omega C_0} \frac{K^2}{1 + K^2}
\]

\( X \) = ratio of acoustic impedances \( Z_2, Z_1 \) in media 2 and 1

\( \xi = \exp (2j k_2 x_2) \).

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B. The Input Impedance as a Function of Frequency

The expression for the input impedance (8) seems rather complicated. One can notice, however, that it contains three expressions \( c_i \), \( \gamma_i \), and \( \omega_i \) of identical mathematical structure, and that a variation of the frequency affects different terms entirely differently.

The terms \( l_0 = l_0(K, C_0, \theta_0) \), \( c_i = c_i(\xi_0, \theta_0) \), \( \gamma_i = \gamma_i(\xi_0, \theta_0) \), and \( w_i = w_i(\xi_0, \theta_0, \Gamma, \phi) \) which are completely defined in terms of basic delay line and transducer parameters, show a significant difference.

The expressions for \( l_0, c_i, \gamma_i \) contain only

\[
\theta_0 = k_{00} x_1 = \frac{\omega}{v_1} x_1 = \omega \tau_1,
\]

while \( w_i \) contains, in addition, the phase angle

\[
\Phi = 2k_{00} x_2 = 2\omega \tau_2.
\]

\( \tau_1 \) and \( \tau_2 \) are the travel times of the acoustic waves through media 1 and 2, respectively. Since \( x_2 \) is several orders of magnitude in excess of the transducer thickness \( x_1 \) one can state that \( \tau_2 \gg \tau_1 \). Therefore a frequency change which causes \( \phi \) to increase by \( 2\pi \) leaves the other terms essentially unchanged. Obviously, (8) can be substantially simplified if one succeeds in separating the “slow moving” terms which contain \( \theta_0 \) from the “fast moving” terms, containing \( \Phi \).

C. Linearization of the Expression for the Input Impedance

A transformation which separates the “slow moving” terms from the “fast moving” is derived in Appendix 1. Using (56) one finds for \( i = 0, 1, 2 \),

\[
c_i[\exp(j\omega w_i)] = F_i + G_i \exp(j\phi_i)
\]

with

\[
F_i = c_i[\exp(j\omega w_i)]
\]

\[
G_i = c_i \rho_i
\]

\[
L_i = \Gamma \rho_i
\]

\[
\rho_i = \frac{2\Gamma}{1 - \Gamma^2} \cos \gamma_i
\]

\[
\psi_i = \Phi + \Delta_i + \eta(\Phi + \Delta_i)
\]

\[
\Delta_0 = \Delta_2 = 2\alpha_1, \quad \Delta_1 = \pi - 2\alpha_0
\]

\( \text{Equations (50) through (68) are located in the Appendixes.} \)
with

$$
\eta(\Phi + \Delta \phi) = \arcsin \frac{2\Gamma}{1 - \Gamma^2} \sin (\Phi + \Delta \phi) \left[ 1 + \Gamma \cos (\Phi + \Delta \phi) + \frac{2\Gamma^2}{1 - \Gamma^2} \sin^2 (\Phi + \Delta \phi) \right] \frac{1 + \left( \frac{2\Gamma}{1 - \Gamma^2} \right)^2 \sin^2 (\Phi + \Delta \phi)}{1 + \left( \frac{2\Gamma}{1 - \Gamma^2} \right)^2 \sin^2 (\Phi + \Delta \phi)}.
$$

Therefore

$$
Z = Z_s + \frac{1}{j\omega C_0} + l_0F + l_0Ge^{j\phi_0} + \delta l_0He^{j\phi_0}
$$

and finally,

$$
Z = Z_s + \frac{1}{j\omega C_0} + l_0F + l_0Ge^{j\phi_0} + \delta l_0He^{j\phi_0}
$$

where

$$
F = F_0[1 + \delta(F_1 - F_2 - j)] = F(\xi_0, \Gamma, \theta, \delta)
$$

is independent of \( \Phi \),

$$
G = G_0[1 + \delta(F_1 - F_2 - j)] = G(\xi_0, \Gamma, \theta, \delta)
$$

is also independent of \( \Phi \),

$$
H = (F_0e^{-j\phi_0} + G_0)(G_0e^{j\phi_0} - G_0e^{j\phi_0})
$$

is \( H(\xi_0, \Gamma, \theta, \Phi) \) a function of \( \Phi \).

The separation of the “slow moving” terms from the “fast moving” terms, \( \psi(\Phi) \) and \( H(\Phi) \), has thus been accomplished.

**D. Interpretation**

The expression for the input impedance of a single-ended delay line (13) can be interpreted as a sum of a number of individual “slow moving” and “fast moving” terms which can best be represented in the impedance plane as shown in Fig. 2.

The symmetry of the circular impedance locus is removed by the small perturbation term \( \delta l_0H \). Since \( H \) depends on \( \Phi \) as indicated by (16), it must be categorized as a “fast moving” term whose magnitude and direction (in respect to the radius vector) varies within one single sweep around the impedance circle. This perturbation, however, exists only for \( \delta \neq 0 \), which is in the case of a lossy transducer. Indeed, small perturbations of the impedance circle have been observed experimentally (see Fig. 5).

The significance of the distortion of the circular impedance locus can be seen in the fact that this perturbation is directly proportional to the loss within the transducer. Thus, a study of this distortion and its phase opens up the possibility of estimating the loss in medium independently. This will be attempted in another paper.

The manner in which the radius vector sweeps the circle as a function of \( \Phi \) is described by \( \phi_0(1') \) and \( H(1') \), has thus been accomplished.

**Since**

$$
\lim_{\xi \to -\infty} \Gamma = 0, \quad \lim_{\xi \to -\infty} G = 0, \quad \lim_{\xi \to -\infty} H = 0
$$

one finds readily the expression for the iconocenter

$$
Z_I = Z_s + \frac{1}{j\omega C_0} + l_0F + l_0Ge^{j\phi_0}[1 + \delta(c_1e^{\gamma_1} - c_2e^{\gamma_2} - j)].
$$
E. Determination of the Material's Loss and the Sound Velocity in the Delay Medium

The following analysis is based on the relation between the angle $\psi_0$ (see Fig. 2) and the frequency $f$. According to one of the equations following (8),

$$\Phi = 2k_0x_2 = 2 \frac{\omega}{v_2} x_2 = 4\pi r_2 f.$$  \hfill (19)

According to (63),

$$\psi_0(\Phi^*_0) = \Phi^*_0 + \eta(\Phi^*_0) \quad \text{with} \quad \Phi^*_0 = \Phi + \Delta \Phi.$$

Since $\eta = 0$ for $\Phi^*_0 = n\pi, n = 0, \pm 1, \pm 2, \cdots$, one can see that $\psi_0(n\pi) = \psi_0$. A plot of $\psi_0$ versus $\Phi^*_0$ is shown in Fig. 3.

The slopes $S_1$ and $S_2$ of $\psi_0$ at the points, corresponding to $\Phi^*_0 = 2n\pi$ and $\Phi^*_0 = (2n - 1)\pi$ are

$$S_1 = \frac{d\psi_0}{d\Phi^*_0} |_{\Phi^*_0=2n\pi} = 1 + \frac{d\eta}{d\Phi^*_0} |_{\Phi^*_0=2n\pi} = \frac{1 + \Gamma}{1 - \Gamma},$$

and

$$S_2 = \frac{d\psi_0}{d\Phi^*_0} |_{\Phi^*_0=(2n-1)\pi} = 1 + \frac{d\eta}{d\Phi^*_0} |_{\Phi^*_0=(2n-1)\pi} = \frac{1 - \Gamma}{1 + \Gamma} = S_1^{-1}.$$  \hfill (20)

$S_1$ and $S_2$ refer to the maximum and minimum slopes, respectively, because

$$\frac{d^2\psi_0}{d\Phi^*_0^2} |_{\Phi^*_0=n\pi} = \frac{d^2\eta}{d\Phi^*_0^2} |_{\Phi^*_0=n\pi} = 0.$$  \hfill (21)

The experimental data are usually available as a function of frequency $\psi_0 = \psi_0(f)$ as shown in Fig. 10 (see page 952). By taking the ratio $S_1/S_2$ any arbitrary scale factor $m$ can readily be eliminated. This leads to the value of $\Gamma$ in a straight-forward way:

$$\frac{S_1}{S_2} = \frac{mS_1}{mS_2} = \left(\frac{1 + \Gamma}{1 - \Gamma}\right)^2;$$

$$\Gamma = \frac{[S_1/S_2]^{1/2} - 1}{[S_1/S_2]^{1/2} + 1}.  \hfill (22)$$

Since a change of $\Phi^*_0$ (or $\Phi$) by $2\pi$ is equivalent, according to (19), to a frequency change

$$\Delta f = (2\tau_2)^{-1},$$

one obtains the round-trip time $(2\tau_2)$ from the readily measured "repetition frequency" $\Delta f$ without resorting to a pulse-echo measurement. With $x_2$ for the length of the delay medium, the sound velocity becomes

$$v_2 = \frac{2x_2}{2\tau_2} = 2x_2\Delta f.$$  \hfill (23)

For a precise measurement of the materials loss, it is desirable to make an assessment of the error which is caused by the neglect of the two angular perturbation terms $\epsilon$ and $\epsilon_H$, which can be obtained from (13), (15), and (16) as

$$\epsilon = \frac{\text{Im} \{G\}}{\text{Re} \{G\}} \quad \text{and} \quad \epsilon_H = \delta \frac{\text{Im} \{H\}}{\text{Re} \{G\}}.$$  \hfill (24)

The "slow moving" term $\epsilon$ which merely shifts the $\psi_0$-scale by $-\epsilon$ has no influence. The term $\epsilon_H$, however, as a "fast moving" term may change $S_1$ and $S_2$ slightly. Its contribution can be determined and, if necessary, subtracted out by calculating $d\epsilon_H/d\psi_0^*$ at points $2n\pi$ and $(2n-1)\pi$.

The unperturbed slopes can then be obtained from the experimentally measured slopes in the form

$$S_{1,2} = \left(\frac{d(\psi_0 + \epsilon + \epsilon_H)}{d\psi_0^*}ight)_{1,2} - \left(\frac{d\epsilon_H}{d\psi_0^*}ight)_{1,2}.  \hfill (25)$$

It is important to note that the series impedance $Z_s$ does not enter this calculation. The proposed procedure makes it possible, therefore, to determine the loss in the delay-line material completely independent of the impedance due to the electrode structure!

F. Relations Between Delay-Line Parameters and Impedance Locus for the Case $\delta=0$

Since the observed deviation from a circular impedance locus is rather small in practice (as is evident in Fig. 5) one may be justified, for the sake of simplicity, to develop the explicit relations between impedance locus and delay line parameters for $\delta=0$. In this case (12) simplifies to

$$Z = Z_s + \frac{1}{j\omega C_0} + l_0 F_0 + l_0 G_0 e^{i\delta_0}.  \hfill (26)$$

The representation of $Z$ in the impedance plane is almost the same as shown in Fig. 2 with some modifications: $F \rightarrow F_0$, $G \rightarrow G_0$, the perturbation term $\delta_0 H$ and the angular perturbation of $\psi_0$ are now missing. With

$$Z_s = R_s + jX_s,$$  \hfill (27)
and the use of (10) and (11), (29) can be written in the form
\[ Z = [R_o + l_0 \omega_0 (\cos \gamma_0 + L_o)] 
+ j[X_o - (\omega_0 C_o)^{-1} + l_0 \omega_0 \sin \gamma_0] + l_0 \omega_0 \phi e^{j \phi}(31)\]
or
\[ Z = R_o + jX_o + P_o e^{j \phi}. \quad (32)\]

In this presentation the impedance circle is characterized by the coordinates of the center \( R_o, X_o \) and the radius \( P_o \). By inserting the appropriate expressions for the basic parameters, one finds
\[ R_o = R_s + \frac{1 + \Gamma^2}{2 \Gamma} P_o \quad (33)\]
\[ X_o = X_s - \frac{1}{\omega_0 C_o} \left[ 1 - \frac{K^2}{1 + K^2} \Omega_1(\theta_0, \xi_0) \right] \quad (34)\]
\[ P_o = \frac{2 \Gamma}{1 - \Gamma^2} \frac{1}{\omega_0 C_o} \frac{K^2}{1 + K^2} \Omega_2(\theta_0, \xi_0) \quad (35)\]

where
\[ \Omega_1(\theta_0, \xi_0) = \frac{c_0 \cos \gamma_0}{\theta_0} = \frac{(1 - \cos \theta_0)^2}{\theta_0} \frac{\xi_0}{\xi_0^2 \cos^2 \theta_0 + \sin^2 \theta_0} \quad (36)\]
and
\[ \Omega_2(\theta_0, \xi_0) = \frac{c_0 \sin \gamma_0}{\theta_0} = -\frac{\sin \theta_0}{\theta_0} \frac{2 - (2 - \xi_0^2) \cos \theta_0}{\xi_0^2 \cos^2 \theta_0 + \sin^2 \theta_0} \quad (37)\]

The characteristic quantities of the impedance locus \((R_o, X_o, P_o)\) of a single-ended delay line have, therefore, been expressed explicitly in terms of the essential delay-line parameters.

Here the acoustic impedance ratio \(\xi_0\) and the phase factor \(\theta_0 = \pi(f/f_o)\) are concentrated in the expressions \(\Omega_1\) and \(\Omega_2\). (\(\Omega_1\) is shown in Fig. 6 as a function of \(\theta_0\) with \(\xi_0\) as a parameter.) The other parameters are the materials loss \(\Gamma\) of the delay medium, the effective electromechanical coupling coefficient \(K\) of the transducer, the transducer capacity \(C_o\), and the real and imaginary part \(R_s, X_s\) of the series impedance caused by the electrode structure. In the independent equations, (33), (34), and (35), only the characteristic quantities \((R_o, X_o, P_o)\) of the impedance locus have been used. If in addition a measurement of \(\phi(f)\) is carried out, additional independent equations can be obtained. When, for instance, \(S_1\) and \(S_2\) are the maximum and minimum of \(d\phi/df\), one obtains the materials loss \(\Gamma\), as described in Section II-E, in the form
\[ \Gamma = \frac{[S_1/S_2]^{1/2} - 1}{[S_1/S_2]^{1/2} + 1}. \quad (24)\]
The location of the iconocenter in the impedance plane, which describes the input impedance for pulsed operation, can be obtained from (31) by setting \(L_o = \rho_o = 0\). With
\[ Z_J = R_J + jX_J \quad (38)\]
one finds
\[ R_J = R_s + \frac{K^2}{1 + K^2} \frac{1}{\omega_0 C_o} \Omega_1 \quad (39)\]
\[ X_J = X_o - \frac{1}{\omega_0 C_o} \left[ 1 - \frac{K^2}{1 + K^2} \Omega_2 \right] \quad (40)\]

In case the impedance locus is obtained experimentally in a polar or Smith chart presentation, the representative values \(\{\Gamma_0, \mu, \rho_0\}\) (as shown in Fig. 16) can readily be transformed into \((R_o, X_o, P_o)\) as derived in Appendix II.

III. EXPERIMENT

A. General Considerations

In the previous section it was shown how, in the case of a single-ended delay line, input impedance and phase angles are related to materials constants and dimensions of delay line and transducer.

A measurement of the input impedance as a function of frequency is therefore all that is required for this interpretation, independent of the way in which the data are acquired. Therefore, one has a choice of several measuring techniques. In the present experiments (a CdS shear transducer attached to a fused quartz delay line under a c-axis angle of (32 ± 3)° with the substrate) the impedance locus was obtained with a network analyzer and the results were represented on an oscilloscope screen with a polar chart scale and photographed, as shown in Fig. 5.

The sample holder was simply a 50 ohm coaxial cable whose center conductor and outer conductor were connected at the reference plane at \(x = 0\) (see Fig. 1), with the metal electrode on top of the transducer and the other electrode attached to the delay medium, respectively. The position of the short was obtained by replacing the delay line with a metal plate, while the position of the open was obtained by opening up the contacts between sample holder and delay line. The position of the short on the polar chart is shown in Fig. 5.

The experimental results are represented by a series of impedance loci on a Smith chart at frequencies between 70 and 200 MHz, shown in Fig. 4(a), and in evaluated form in the impedance plane in Fig. 4(b). These loci are circles at first glance and only under close scrutiny can one detect a deviation. The example of the impedance locus for a center frequency of 125 MHz, is shown in Fig. 5. The distortions from the circular form are quite small and a restriction to the case of \(\delta = 0\) therefore seems appropriate in the following calculations.

B. Data Evaluation

From the photographs of impedance circles at different frequencies, one obtains values for \(\{\Gamma_0, \mu, \rho_0\}\) which can be transformed to \((R_o, X_o, P_o)\) according to equations derived
Fig. 4. (a) Impedance loci of single-ended delay line for different frequencies on a Smith chart. (b) Loci of input impedance of single-ended delay line in impedance plane for different center frequencies.
Fig. 5. Oscilloscope screen showing input impedance locus on a polar chart at 125 MHz indicating slight deviation from circular form.

The results are shown in Table I. These raw data can now be evaluated in a variety of ways. One particular way is pursued below.

1) Determination of \( \xi_0 \) and \( K \): Equations (34) and (35) yield the following expression:

\[
\left( \frac{P_0}{-(X_0 - X_0)} \right) \left( \frac{1 - \Gamma^2}{2\Gamma} \right) = \frac{k^2\Omega_1}{1 - k^2\Omega_2} \tag{41}
\]

where

\[
k^2 = \frac{K^2}{1 + K^2}.
\]

If one makes the assumption that \( X \ll X_0 \) and \( k \ll 1 \), one can obtain

\[
Q = \frac{P_0}{-X_0} \frac{1 - \Gamma^2}{2\Gamma} = k^2\Omega_1 \left[ 1 + k^2\Omega_2 + \frac{X_1}{X_0} \right] \tag{42}
\]

or to a cruder approximation

\[
k^2 = Q \frac{\Omega_1}{\Omega_1}. \tag{43}
\]

Since \( P_0 \) and \( X_0 \) are known from experiment as a function of frequency, \( Q \) can be determined as soon as the materials loss \( \Gamma \) is known. In the following evaluation, \( \Gamma \)-values for shear waves in a delay line of fused quartz and a length \( x_s = 1.27 \) cm have been taken from the work of Fraser et al.\(^4\)

With these \( \Gamma \)-values (Table I) \( Q(f) \) has been calculated. The result is shown in Fig. 7.

It is now quite instructive to compare the “shape” of \( Q(f) \) with the shape of \( \Omega_1 \), which according to Fig. 6 changes significantly as a function of \( \xi_0 \).

In order to define “shape” two simple criteria are introduced which assess two different features of the two curves to be compared.

Fig. 6. Function $\Omega(\xi_0, \theta_0)$ for different values of the parameters $\xi_0$.

Fig. 7. $Q(f)$, (42), and shape criteria calculated from experimental data ($X_0, P_0$).
their maximum value. Then the two shapes are equal when their normalized widths are equal,
\[
W_{\Omega^*}(\xi_0) = W_{\Omega}(\xi_0)/\theta_m(\xi_0) \\
W_Q^* = W_Q/f_m.
\]
(45)
The application of this second criterion to \( Q(f) \), where \( W_Q^* = 0.30 \), leads to a value of \( \xi_0 = 2.25 \), as shown in the lower part of Fig. 8.
Both "shape" criteria, although addressing themselves to different features of the "shape," agree rather well and indicate that \( \xi_0 \) should be approximately 2.2 as indicated by a dashed vertical line in Fig. 8. According to (43), the electromechanical coupling coefficient becomes to a first approximation
\[
K = \left[ k^2/(1 - k^2) \right]^{1/2} \quad \text{with} \quad k^2 = \frac{Q(f)}{\Omega_\Omega(\theta_0, 2.2)}. 
\]
(46)
The relation between \( f \) and \( \theta_0 \) is established by
\[
\theta_0 = \frac{\theta_m}{f_m}. 
\]
(47)
The calculation is shown in Table II. The resulting value for \( K \) is
\[
K = (0.179 \pm 0.001). 
\]
The indicated accuracy is the probable error. One has to keep in mind, however, that this result needs to be corrected with terms exhibited in (42).

2) Determination of the Series Resistance \( R_s \) and Series Reactance \( X_s \): Using the experimental values for \( R_0 \) and \( P_0 \) and the literature value for \( \Gamma \), one obtains from (33) the frequency dependence of \( R_\Omega \) as shown in Fig. 9. The drop of \( R_\Omega \) with increasing frequency is consistent with the fact that the series resistance \( R_s \) is actually of a distributed nature.
TABLE II
Calculation of the Electromechanical Coupling Coefficient and of the Capacity of the Transducer

<table>
<thead>
<tr>
<th>( f ) (MHz)</th>
<th>( \theta_f = \theta_m f )</th>
<th>( \Omega_1(\theta_m, 2.2) )</th>
<th>( \Omega_2(\theta_m, 2.2) )</th>
<th>( k^2 = \frac{Q}{Q_1} )</th>
<th>( C_o \approx \frac{1-k^2\Omega_1}{-100\pi fX_o} ) (pF)</th>
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<td>70</td>
<td>49.97</td>
<td>0.124</td>
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<td>3.09</td>
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</tr>
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<td>160</td>
<td>114.21</td>
<td>1.327</td>
<td>-0.220</td>
<td>2.99</td>
<td>46.4</td>
</tr>
<tr>
<td>170</td>
<td>121.35</td>
<td>1.174</td>
<td>-0.100</td>
<td>2.99</td>
<td>46.7</td>
</tr>
</tbody>
</table>

\[ k^2 = (3.10 \pm 0.02) \times 10^{-9} \]
\[ k = (0.179 \pm 0.001) \]
\[ C_o \approx 49.4 \text{ pF} \]

Using (34), it follows that

\[
C_o = \frac{1 - k^2\Omega_2}{-\omega X_0} \left[ 1 + \frac{X_s}{X_0} \right] \text{[for } X_s \ll X_0],
\]

or approximately

\[
C_o \approx \frac{1 - k^2\Omega_2}{-\omega X_0}. \tag{49}
\]

The calculation is shown in Table II.

C. Independent Determination of the Sound Velocity and the Materials Loss

In a point by point measurement, the angle \( \varphi(f) \) of the radius vector in the impedance plane was obtained for several center frequencies. The result of such a measurement is shown in Fig. 10 for a center frequency of 125 MHz. The repetition frequency, that is, the frequency change necessary to complete a circle in the impedance plane, is \( \Delta f = 148 \text{ kHz} \).

According to (25), the round-trip time is then 6.76 ms in excellent agreement with the value obtained by pulse-echo technique.

Since the delay line has a length of 1.27 cm, the sound velocity according to (26), becomes

\[ v_s = 3.76 \times 10^5 \text{ cm/s}. \]

This value agrees well with the published data for shear waves in fused quartz.\(^5\)

The slopes \( mS_1 \) and \( mS_2 \) as indicated in Fig. 10 can be determined quite accurately and lead to a value of \( \Gamma = 0.685 \) at 125 MHz, according to (24). This value corresponds to a round-trip loss of 3.3 dB or a propagation loss of \( \alpha = 1.3 \) dB/cm. The \( \Gamma \) values measured at different frequencies are collected in Fig. 11. The connecting curve follows the general trend of increasing loss with increasing frequency.

In order to compare the measured values with those of Fraser et al.,\(^4\) one must take into account the diffraction loss. A calculation of the diffraction loss\(^6\) was made for the present configuration, and is shown in Fig. 11 as a dashed line. When the diffraction loss is subtracted from the measured loss, a “corrected materials loss” emerges which is to be compared to the materials loss obtained by Fraser et al. The comparison shows a discrepancy of a fraction of a decibel at the lower frequency and becomes increasingly severe at higher frequencies as shown in Fig. 11.

---

Fig. 10. Angular dependence \( \psi(\phi) \) for a center frequency of 125 MHz.

Fig. 11. Measured materials loss and diffraction loss of single-ended delay line and acoustic loss for shear waves in quartz.
IV. DISCUSSION

By comparing the measured and the expected materials loss for fused quartz, it turns out that at low frequency the discrepancy between these values is rather small and is within the expected fluctuations which can show up from material to material. Since there is reasonably good agreement at the lower frequency end, it is quite strange that the discrepancy should grow so drastically with increasing frequency.

Another discrepancy has been encountered earlier in Section III-B1 where it was found that the ratio of acoustic impedances for shear waves in fused quartz and in CdS should have the value $\xi_0 = 2.2$. This result is in striking disagreement with the expected value of $\xi_0 = 0.99$ for this impedance ratio.

One may follow a hunch that these two discrepancies mentioned above are possibly related. In setting up the mathematical model it was assumed, for reasons of mathematical economy, that the electrode structure is noticeable only through its electrical effects represented by a series impedance $Z_s'$. This notion can no longer be held up and it must be assumed that for this particular delay line under investigation, the electrode structure has also an effect acoustically:

1) the passage of the soundwave through the electrode layer is accompanied by an additional loss which increases markedly at shorter wavelengths;

2) the electrode structure of this particular device provides for a barrier which leads to a pronounced reflection of shear waves at the interface between medium 1 and 2, contrary to the small acoustical impedance mismatch which would be expected at the interface between the two unperturbed materials.

These findings suggest that the present mathematical model should be expanded and refined to accommodate the acoustical effects of the electrode structure.

A similarity of the present technique with the nodal shift method is suggested by Fig. 3 and Fig. 10. Instead of moving an electrical short (or open) along an output terminal, one can imagine that an acoustical short (or open) is moved instead at the end of the acoustical delay line. This effect can be easily achieved by a small frequency change which leaves the “slow moving” parameters unaltered.

One may ask the question why the analysis in the experimental section was carried through only for the case $\delta = 0$. The reason is that it was felt that the accuracy of the data obtained with the present instrumentation were not yet sufficient to warrant a full effort at this time. However, an estimate was undertaken, using the deviation from the circular form, as shown in Fig. 5, to yield a value for $\delta$, which is equivalent to an attenuation $\alpha_f \approx 20$ dB/cm at 125 MHz. This is indeed the order of magnitude for single crystalline CdS which is observed for the attenuation of shear waves at this frequency. It also confirms that the assumptions made in (7) are justified.

APPENDIX I

In (8) three expressions, $w_0$, $w_1$, $w_2$ of identical mathematical structure appear. The analysis of one, for instance $w_0$, leads to a linearization which allows further analytical pursuit and simple geometrical interpretation of (8). The complex function $w_0$ can be expressed in the form

$$w_0 = \frac{z_1}{z_2} = \frac{1 + \Gamma \exp \left[ j(\Phi - 2\alpha_0) \right]}{1 - \Gamma \exp \left[ j(\Phi + 2\alpha_1) \right]}$$

where

$$\Phi' = \Phi - (\frac{3}{2} \pi + \gamma_0 - 2\alpha_1)$$

$$\beta = \frac{3}{2} \pi - \gamma_0$$

$$\gamma_0 = \alpha_0 + \alpha_1.$$
\[ \theta_0' = \pi \]
\[ w_{03} = \exp \left(j2\theta_1 \right) \]
\[ \tan \theta_3 = (-q)/(1-h) \]
\[ \theta_0' = \pi + \arccos h \]
\[ w_{04} = w_{02}^{-1} \]
\[ \theta_4 = 0. \]

The transformation (50) is a conform mapping. Therefore, the circle \( K_0 \) is transformed into a circle \( K_0' \). The position of the points \( w_{01} \) to \( w_{04} \) are known, as indicated in Fig. 13. We are therefore entitled to describe \( K_0' \) in the form

\[ w_0 = 1 + L_0 e^{j\theta} + \rho_0 e^{j(\Phi - \gamma_0)} \]  

B. Determination of the Radius and the Position of the Circle \( K_0' \)

\[ x_0 = \frac{1}{2} (w_{02} + w_{04}) = 1 + 2q^2/p(p + 2q) \]
\[ y_0 = x_0 \tan (\theta_1 + \theta_2) = (-2qh)/p(p + 2q) \]
\[ \rho_0 = \sqrt{(w_{04} - x_0)^2 + y_0^2} = 2q/p(p + 2q) \]
\[ = \frac{2\Gamma}{1 - \Gamma^2} \cos \gamma_0 \]  

\[ L_0 = \sqrt{(x_0 - 1)^2 + y_0^2} = 2q\Gamma/p(p + 2q) = \Gamma \rho_0 \]
\[ \tan \xi = y_0/(x_0 - 1) = (-h)/q = - \tan \gamma_0 \]
\[ = \tan (-\gamma_0) \rightarrow \xi = - \gamma_0. \]

Therefore

\[ e^{j\gamma_0 w_0} = e^{j\gamma_0} + L_0 + \rho_0 e^{j(\Phi - \gamma_0)}. \]

In general

\[ e^{j\gamma_0 w_i} = e^{j\gamma_0} + L_i + \rho_i e^{j(\Phi - \gamma_0)} \]  

C. Determination of the Relation Between \( \psi_i \) and \( \Phi \)

We shall now calculate how the angle \( \psi_i \) in the w-plane is related to the angle \( \Phi \) in the Z-plane. This is shown in Fig. 14. In order to see the relation between Fig. 12 and Fig. 14, one has to keep in mind that Fig. 14 is the same as Fig. 13 but rotated by the angle \( \gamma_0 \) around the origin. The center \( C \) of the circle \( K_0' \) and the straight line \( g \) through the center in Fig. 12 are transformed into point \( C' \) and circle \( g' \) in Fig. 14. The straight line \( t' \) is tangent to \( g' \) at \( C' \). These few explanations, together with the notion that a transformation by (50) preserves angles, should suffice to make Fig. 14 intelligible.

We now write \( \psi_i \) as the sum of the original angle \( \Phi \) and a correction term \( \eta \) and allow for an additive term \( \Delta \).

Assume

\[ \psi_i(\Phi^*) = \Phi^* + \eta(\Phi^*) \quad (i = 0, 1, 2) \]

\[ \Phi^* = \Phi + \Delta \iota. \]

Dropping the index \( i \) for the moment,

\[ \psi(\Phi^*) = \Phi^* + \eta(\Phi^*), \]
as noted in Fig. 14. The following relations can be confirmed by inspection of Fig. 14:

\[
L / \sin \eta_a = R / \sin \left( \frac{\pi}{2} + \Phi^* \right) \quad \rightarrow \quad \sin \eta_a = (L/R) \cos \Phi^*
\]

\[
\sin \eta(\Phi_i^*) = \frac{[2\Gamma/(1 - \Gamma^2)] \sin \Phi_i^* \left[ 1 + \Gamma \cos \Phi_i^* + [2\Gamma^2/(1 - \Gamma^2)] \sin^2 \Phi_i^* \right]}{1 + [2\Gamma/(1 - \Gamma^2)] \sin^2 \Phi_i^*}
\]

where the index \( i \) has been returned. However, the additive term \( \Delta_i \) has yet to be determined.

**D. Relation Between \( \Phi \) and \( \Phi_i^* \)**

In order to determine \( \Delta_i \), we consider special case ①. There \( \Phi_0 = 0 \). Therefore, according to (51),

\[
\Phi = \frac{\pi}{2} + \gamma_0 - 2\alpha_1.
\]

According to Fig. 15

\[
\Phi_0^* = \frac{\pi}{2} + \gamma_0.
\]

Therefore

\[
\Delta_0 = \Phi_0^* - \Phi = 2\alpha_1.
\]

In a similar way one finds

\[
\Delta_1 = \pi - 2\alpha_0 \quad \text{and} \quad \Delta_2 = 2\alpha_1 = \Delta_0.
\]
APPENDIX II

The complex reflection coefficient \( \Gamma \) in the Smith chart can be translated into a point \( \xi \) in the normalized impedance plane by the transformation

\[
\Gamma = \frac{\xi - 1}{\xi + 1}.
\]

(67)

A circle on the Smith chart, as shown in Fig. 16, with center \( \Gamma_0 = |\Gamma_0| e^{j\theta} \) and radius \( r_0 \) can then be described in the form

\[
\Gamma(\phi) = \frac{\xi(\phi) - 1}{\xi(\phi) + 1} = \Gamma_0 + r_0 e^{j\phi}. \tag{68}
\]

The transformation \( \Gamma(\phi) \rightarrow \xi(\phi) \) can now be broken up into a series of small steps which lend themselves to easy geometrical interpretation:

\[
\begin{align*}
\Gamma_1 &= \Gamma^{-1} = (\xi + 1)/(\xi - 1) \\
&= \left[\frac{1}{2}(\xi - 1)\right]^{-1} + 1
\end{align*}
\]

\[
\begin{align*}
\Gamma_2 &= \Gamma^{-1} - 1 \\
&= \left[\frac{1}{2}(\xi - 1)\right]^{-1} - 1
\end{align*}
\]

\[
\begin{align*}
\Gamma_3 &= \frac{1}{2}(\Gamma^{-1} - 1) \\
&= (\xi - 1)^{-1}
\end{align*}
\]

\[
\begin{align*}
\Gamma_4 &= \left[\frac{1}{2}(\Gamma^{-1} - 1)\right]^{-1} = \xi - 1 \\
\Gamma_5 &= \left[\frac{1}{2}(\Gamma^{-1} - 1)\right]^{-1} + 1 = \xi
\end{align*}
\]

Therefore

\[
\begin{align*}
R_0 &= 1 - |\Gamma_4| (1 - |\Gamma_1| \cos \mu)/|\Gamma_2| \\
X_0 &= |\Gamma_4| \sin \delta = |\Gamma_4| (|\Gamma_1| \sin \mu)/|\Gamma_2| \\
P_0 &= r_4.
\end{align*}
\]

By process of elimination, one arrives at

\[
\begin{align*}
R_0 &= (1 - Y_0^2)/D \\
X_0 &= 2 |\Gamma_0| (\sin \mu)/D \\
P_0 &= 2r_0/D
\end{align*}
\]

\[
D = 1 + Y_0^2 - 2 |\Gamma_0| \cos \mu
\]

\[
Y_0^2 = \Gamma_0^2 - r_0^2.
\]

This result has an obvious geometrical interpretation. From quantities \( N, M, Y \) as shown in Fig. 17, one obtains

\[
R_0 = M^2 Y^{-2}, \quad X_0 = 2N Y^{-2}, \quad P_0 = 2r_0 Y^{-2}.
\]