Excitation of Elastic Waves in Crystals

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Abstract—Excitation of elastic waves in a general anisotropic crystal is discussed. The crystal is supposed to be semi-infinite, bounded by a plane surface. The definition of an elastic impedance matrix for the medium and its use in the subsequent discussion emphasize the similarities with transmission line theory. The resulting expressions for the amplitudes of bulk waves and surface waves is expressed in a matrix formalism believed to have computational advantages.

I. Introduction

The excitation of elastic waves in crystals has been studied by a number of authors [1]–[8]. Buchwald [1] considers the excitation of surface waves from a point source in a general anisotropic medium. His treatment stresses the importance of the geometrical form of the wave-vector surface (slowness surface) of the medium. In Buchwald’s work the case of a medium with transverse anisotropy has received special attention. Viktorov [2] has considered, in particular, the excitation from various surface wave transducers for an isotropic medium. In addition to surface waves, Viktorov also considers the excitation of bulk transverse waves with a direction of propagation parallel with the surface. More recently, the excitation of surface waves in a piezoelectric medium from interdigital surface electrodes has received much attention [3]–[8].

The present paper is primarily concerned with elastic waves in a semi-infinite medium. The particular notation used has been chosen to facilitate the solution of boundary value problems for plane boundaries. The concept of an elastic impedance matrix is introduced and its use tends to emphasize the connection between transmission line theory and the problem of reflecting elastic waves in a plane boundary of a general crystal. The phenomenon of surface waves appears in this treatment as a result of singularities in the impedance matrix. The formalism and concepts that are used have been developed in greater detail elsewhere [9]–[10]. A brief summary is given here.

We shall consider plane waves with wave vectors having a given projection \( \mathbf{q} = (q_x, q_z) \) in the \( xz \) plane. The \( xz \) plane will later be taken to be the boundary of a semi-infinite medium, but at the outset the medium is considered infinite.

Rather than working directly with the stress tensor \( \sigma_{ij} \), we shall use the stress on a plane parallel with the \( xz \) plane and introduce the three-dimensional column vector

\[
\mathbf{v} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ \sigma_{zz} \end{bmatrix}.
\] (1)

The displacement is denoted by the three-dimensional column vector

\[
\xi = \begin{bmatrix} \xi_x \\ \xi_y \\ \xi_z \end{bmatrix}.
\] (2)

In order to introduce a suitable matrix formalism we also define a six-dimensional column vector \( \xi \) consisting of a combination of \( \xi \) and \( \mathbf{v} \):

\[
\xi = \begin{bmatrix} \xi \\ \mathbf{v} \end{bmatrix}.
\] (3)

We shall consider time-harmonic, acoustic fields of frequency \( \omega/2\pi \) that have also a harmonic variation with respect to \( x \) and \( z \). The column vectors introduced in (1)–(3) are complex amplitude vectors that are functions of \( y, q_x, q_z \), and \( \omega \). The displacement written in full as function of the space coordinates \( x, z \), and time is

\[
\Re \{ \xi e^{-i(\omega t-q_x x-q_z z)} \}.
\] (4)

Let us assume that a volume force of complex amplitude \( \mathbf{f} \) excites elastic waves in the medium. The force is taken to be harmonically varying with frequency \( \omega \) and its amplitude \( \mathbf{f} \) is a function of the coordinates \( x, y, \) and \( z \). We assume that the force is zero outside a finite region near the origin, and on the positive side of the \( xz \) plane. This insures the existence of the Fourier transform

\[
f(y, q_x, q_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{i(q_x x+q_z z)} dx dz.
\] (5)

In terms of the notation introduced, the equation of motion of the medium may be shown to take the form

\[
\frac{d}{dy} \xi + jN\xi = \begin{bmatrix} 0 \\ \mathbf{f} \end{bmatrix}
\] (6)

where \( 0 \) is the three dimensional zero vector, and \( N \) is a \( 6 \times 6 \) matrix with terms that are homogeneous polynomials in \( q_x, q_z \), and \( \omega \) [9], [10]. The coefficients in these polynomials are determined by the elastic tensor of the medium.

The eigenvalues of \( N \) are denoted by

\[
n_1, n_2, \ldots, n_6
\]

and the corresponding eigenvectors by

\[
\xi_1, \xi_2, \ldots, \xi_6.
\]

The eigenvector \( \xi_i \) represents a plane acoustic wave with wave vector

\[
\mathbf{k} = (q_x - n_i, q_z).
\]
The eigenvalue \( n_i \) may be real, in which case \( \xi_i \) corresponds to a plane, uniform wave. If \( n_i \) is complex, \( n_i^* \) is also an eigenvalue. The two eigenvectors corresponding to \( n_i \) and \( n_i^* \) represent nonuniform plane waves, one of them increasing, the other one decreasing exponentially in the \( y \) direction. For any given \( \tilde{q} \) three critical frequencies \( \omega_1, \omega_2, \omega_3 \) may be defined. They partition the real frequency axis into four regions \( R_0, R_1, R_2, R_3 \) in the manner shown in Fig. 1. For each \( \omega \) in region \( R_\mu \), \( N \) has \( p \) real eigenvalues. For \( \omega \) equal to any of the critical frequencies two eigenvalues coincide, i.e., \( N \) is a degenerate matrix.

Of the eigenvectors corresponding to real eigenvalues, one half represent plane waves with a positive \( y \) component of group velocity. The other half represent plane waves with a negative \( y \) component. We shall call them upward and downward waves, respectively.

Let \( \xi_1, \xi_2, \) and \( \xi_3 \) represent upward waves if the corresponding eigenvalues are real. When one or more eigenvalues are complex, \( \xi_1, \xi_2, \) and \( \xi_3 \) also include waves that decrease exponentially in the \( y \) direction.

An acoustic field consisting of upward waves of frequency \( \omega/2\pi \) may now be written

\[
\xi = \sum_{k=1}^{3} c_k e^{-j\omega y} \xi_k
\]

where the \( c_k \) are constants. Splitting (7) into two parts, we obtain

\[
\xi = \sum_{k=1}^{3} c_k e^{-j\omega y} \xi_k
\]

\[
v = \sum_{k=1}^{3} c_k e^{-j\omega y} v_k.
\]

Let us now define the following \( 3 \times 3 \) matrices

\[
X_1 = (\xi_1, \xi_2, \xi_3); \quad Y_1 = (v_1, v_2, v_3)
\]

\[
X_2 = (\xi_4, \xi_5, \xi_6); \quad Y_2 = (v_4, v_5, v_6).
\]

We further define the \( 3 \times 3 \) diagonal matrices

\[
D_1(y) = \text{diag} \left( e^{-j\omega y}, e^{-j\omega y}, e^{-j\omega y} \right)
\]

\[
D_2(y) = \text{diag} \left( e^{-j\omega y}, e^{-j\omega y}, e^{-j\omega y} \right)
\]

and the column vector

\[
c_1 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
\]

In terms of this notation, (8) and (9) may be written

\[
\xi_+ = X_1 D_1(y) c_1
\]

\[
v_+ = Y_1 D_1(y) c_1
\]

where the subscript \( + \) is used to indicate upward waves.

When \( c_1 \) is eliminated between (15) and (16), we obtain a linear relation between force and displacement

\[
v_+ = j\omega Z_1 \xi_+
\]

Fig. 1. Critical frequencies for given \( \tilde{q} \). \( N \) has 0, 2, 4, or 6 real eigenvalues if \( \omega \) is in \( R_0, R_1, R_2, \) or \( R_3 \), respectively.

\[
v_+ = \frac{1}{j\omega} (Z_1 - Z_2)^{-1} v - (Z_1 - Z_2)^{-1} Z_1 \xi
\]

\[
v_- = \frac{1}{j\omega} (Z_1 - Z_2)^{-1} v + (Z_1 - Z_2)^{-1} Z_1 \xi.
\]
II. Transmission Line Relations for Anisotropic Media

Equations (17) and (20) may be interpreted to mean that the matrices \( Z_1 \) and \( Z_2 = -Z_1^T \) are the characteristic impedances of a general medium for upward and downward waves, respectively. Equation (19), relating the displacement vectors for upward waves at two different values of \( y \), shows that the matrix

\[
X_1 D_1(y') D_1^{-1}(y) X_1^{-1} = K_+(y' - y)
\]

acts as a translation operator, much like the simple exponential \( e^{j(y' - y)} \) in transmission-line theory. Similarly the translation operator for downward waves is, from (23), seen to be

\[
X_1 D_2(y') D_2^{-1}(y) X_2^{-1} = K_-(y' - y).
\]

We shall see that the use of these concepts will facilitate a coherent derivation of the various wave modes that exist in homogeneous and layered media.

Consider first a semi-infinite medium bounded by the plane \( y = 0 \) with the \( y \) axis pointing into the medium. Let the displacement vector of the surface be denoted by \( \xi_0 \). The surface force per unit area required to produce this displacement is

\[
\phi = j\omega Z_1 \xi_0.
\]

For particular values of \( \tilde{q} \) and \( \omega \), \( Z_1 \) may be singular so that a nonzero vector \( \xi_0 \) satisfies

\[
Z_1 \xi_0 = 0.
\]

The condition for this is

\[
|Z_1| = 0.
\]

For real valued \( \omega \) and \( \tilde{q} \) the solutions of (31) are surface waves (Rayleigh waves). Solutions for which \( \omega \) or \( \tilde{q} \) are complex represent the so-called leaky surface waves.

Consider next the case in which the plane \( y = 0 \) is the interface between two semi-infinite elastic media \( A \) and \( B \), whose characteristic impedances for upward waves are denoted by \( Z_A \) and \( Z_B \), respectively. If no power is radiated towards the interface from the regions at \( y = \pm \infty \), only upward waves can exist in one of the media, and only downward waves in the other. The surface force acting on medium \( A \) through the bounding surface is now

\[
\phi = j\omega Z_A \xi_0,
\]

while the surface force acting on medium \( B \) is

\[
-j\omega Z_B^T \xi_0.
\]

Continuity of the stress tensor requires that the two are equal:

\[
(Z_A + Z_B^T) \xi_0 = 0
\]

or

\[
|Z_A + Z_B^T| = 0.
\]

The solutions of (35) are the so-called Stoneley waves.

Consider next a crystal plate of thickness \( d \) bounded by the planes \( y = 0 \) and \( y = d \).

In order to satisfy the boundary conditions we must have

\[
\mathbf{v}_+(0) = -\mathbf{v}_-(0).
\]

From (17), (20), (26), and (27) it is found that the translation operators for \( \mathbf{v}_+(y) \) and \( \mathbf{v}_-(0) \) are

\[
ZK_+(y' - y) Z^{-1}
\]

and

\[
Z^T K_-(y' - y) (Z^T)^{-1},
\]

respectively. The subscript of the impedance matrix has here been omitted. The force per unit area at \( y = d \) is then

\[
\mathbf{v}_+(y) + \mathbf{v}_-(y) = ZK_+(d) Z^{-1} \mathbf{v}_+(0)
\]

\[
- Z^T K_-(d) (Z^T)^{-1} \mathbf{v}_+(0).
\]

Boundary conditions require that this force is zero. This leads to

\[
|ZK_+(d) Z^{-1} - Z^T K_-(d) (Z^T)^{-1}| = 0.
\]

In order to discuss more complex problems such as multilayered anisotropic media it is advantageous to introduce the impedance \( W(y) \) at the ordinate \( y \) by

\[
j\omega W(y) \xi = \mathbf{v}.
\]

In contrast to the characteristic impedance, \( W \) is a function of \( y \).

It is now useful to be able to compute the impedance \( W' \) at the plane \( y' \) from the knowledge of the impedance \( W \) at \( y \). Making use of (26)-(29) and (42), we find

\[
W'(y') = \{ZK_+(Z + Z^T)^{-1}(W + Z^T)
\]

\[
+ Z^T K_-(Z + Z^T)^{-1} (W - Z) \}
\]

\[
\cdot \{K_+(Z + Z^T)^{-1}(W + Z^T)
\]

\[
- K_-(Z + Z^T)^{-1} (W - Z) \}^{-1}
\]

where the argument of \( K_+ \) and \( K_- \) is \( y' - y \). This expression is closely analogous to expressions in transmission line theory.

Consider now the case where a semi-infinite medium of characteristic impedance \( Z_A \) has on its surface a layer of thickness \( d \). The characteristic impedance of the layer is \( Z \).

We may then regard the layer as loaded by the impedance \( Z_A \) of the medium. Putting

\[
W = Z_A
\]

in (43) and using

\[
y' - y = -d
\]

as argument of \( K_+ \) and \( K_- \), we obtain the “input impedance” of the free surface of the layer. In order for nontrivial solutions to exist this impedance must be singular, i.e.,

\[
|W'(-d)| = 0.
\]
For a range of values of \( \omega \) and \( q \) in which no power is radiated in the \( y \) direction the impedance matrices are skew-Hermitian. Their eigenvalues are then reactance functions in the sense that they always increase with frequency. This fact greatly facilitates the numerical search for zeros of the eigenvalues and thereby also for solutions of the determinants of equations such as (32) and (46).

### III. EXCITATION OF WAVES IN A SEMI-INFINITE MEDIUM

#### A. The Infinite Medium

We shall first consider the excitation of acoustic volume waves in an infinite medium and later specialize the results to the case of a semi-infinite medium bounded by a plane surface. The waves are excited by a force \( f(y) \) distributed over the volume. As is evident from (5), this force is a function of the three variables \( y, q_x, \) and \( q_z \). For the sake of brevity its functional dependence of \( y \) only is emphasized by the notation. The waves excited by \( f(y) \) may be found by use of standard methods of solving the vector differential equation (6). This has been done elsewhere [10]. Here a less direct method will be used for the purposes of gaining physical understanding and developing concepts that are of use in the following.

Let us consider the contribution to the excitation from the force acting on a thin, plane layer of thickness \( dy' \) at distance \( y' \) from the \( xz \) plane. This layer is moved by the force \( f(y')dy' \). We shall obtain the displacement that results at a distance \( y \) from the \( xz \) plane, assuming \( y > y' \) (see Fig. 2). The displacement \( \xi(y') \) of the layer may be split into upward and downward waves as shown in the Introduction. Clearly, only the former will contribute to the displacement at \( y \). In (26) \( \xi \) and \( v \), respectively, represent displacement and force caused by free waves. When an impressed force \( f(y')dy' \) is also present \( v \) must be replaced by \( v + f(y')dy' \) and (26) is converted into

\[
\xi_+(y') = \frac{1}{j\omega} (Z_1 - Z_2)^{-1}f(y')dy' + \frac{1}{j\omega} (Z_1 - Z_2)^{-1}v(y') - (Z_1 - Z_2)^{-1}Z_2 \xi(y').
\]

Equation (49) gives the contribution to the displacement at \( y \) from the region of the medium below \( y \). In the same way the region above \( y \) is seen to contribute the amount

\[
\xi_-(y) = \frac{1}{j\omega} \int_{-\infty}^{y} X_t D_t(y) D_t^{-1}(y') X_t^{-1} \cdot (Z_1 - Z_2)^{-1}f(y')dy'
\]

which is a displacement caused by downwards waves. Making use of (22), and introducing the column vectors,

\[
\nu_1(y) = \frac{1}{j\omega} \int_{-\infty}^{y} D_{t1}^{-1}(y')X_{t1}^{-1}(Z_1 + Z_1^T)^{-1}f(y')dy',
\]

\[
\nu_2(y) = -\frac{1}{j\omega} \int_{y}^{\infty} D_{t2}^{-1}(y')X_{t2}^{-1}(Z_1 + Z_1^T)^{-1}f(y')dy'.
\]

Equations (49) and (50) take the form

\[
\xi_+(y) = X_t D_t(y) \nu_1(y),
\]

\[
\xi_-(y) = X_t D_t(y) \nu_2(y).
\]

The total displacement caused by the force distribution may now be obtained by computation of the inverse Fourier transform with respect to \( q_x \) and \( q_z \) of \( \xi_+ + \xi_- \). However, since we are primarily interested in results for the semi-infinite medium, we shall discuss first how a plane boundary of the medium influences the results obtained so far.

#### B. The Semi-Infinite Medium

Let the medium be bounded by the plane \( y = 0 \) so that the positive \( y \) axis is pointing into the medium. The main effect of the boundary is to reflect the downward waves incident on the boundary. The downward waves excited by the force \( f(y) \) cause a displacement at the boundary given by

\[
\xi_-(0) = X_0 \nu_2(0).
\]

This displacement may be transformed to the ordinate \( y \) by means of (19) and subsequent integration gives

\[
\xi_+(y) = \frac{1}{j\omega} \int_{0}^{y} X_t D_t(y) D_t^{-1}(y')X_t^{-1}(Z_1 - Z_2)^{-1}f(y')dy'.
\]

We have here assumed that the volume force \( f \) is confined to the region \( y > 0 \) so that the lower limit of integration is zero.

Equation (51) gives the contribution to the displacement at \( y \) from the region of the medium below \( y \). In the same way the region above \( y \) is seen to contribute the amount

\[
\xi_-(y) = \frac{1}{j\omega} \int_{y}^{\infty} X_t D_t(y) D_t^{-1}(y')X_t^{-1} \cdot (Z_1 - Z_2)^{-1}f(y')dy'
\]

which is a displacement caused by downwards waves. Making use of (22), and introducing the column vectors,

\[
\nu_1(y) = \frac{1}{j\omega} \int_{y}^{\infty} D_{t1}^{-1}(y')X_{t1}^{-1}(Z_1 + Z_1^T)^{-1}f(y')dy',
\]

\[
\nu_2(y) = -\frac{1}{j\omega} \int_{y}^{\infty} D_{t2}^{-1}(y')X_{t2}^{-1}(Z_1 + Z_1^T)^{-1}f(y')dy'.
\]

Equations (49) and (50) take the form

\[
\xi_+(y) = X_t D_t(y) \nu_1(y),
\]

\[
\xi_-(y) = X_t D_t(y) \nu_2(y).
\]

The total displacement caused by the force distribution may now be obtained by computation of the inverse Fourier transform with respect to \( q_x \) and \( q_z \) of \( \xi_+ + \xi_- \). However, since we are primarily interested in results for the semi-infinite medium, we shall discuss first how a plane boundary of the medium influences the results obtained so far.

#### B. The Semi-Infinite Medium

Let the medium be bounded by the plane \( y = 0 \) so that the positive \( y \) axis is pointing into the medium. The main effect of the boundary is to reflect the downward waves incident on the boundary. The downward waves excited by the force \( f(y) \) cause a displacement at the boundary given by

\[
\xi_-(0) = X_0 \nu_2(0).
\]

The force associated with this displacement is

\[
j\omega Z \xi_-(0).
\]

The force associated with the reflected waves must be the negative of (56) and the corresponding displacement at the boundary is, from (17), seen to be

\[
- Z_1^{-1} Z \xi_-(0).
\]
We shall suppose that in addition an externally impressed surface force $\phi$ acts on the boundary so that the total displacement is

$$\xi_+(0) = -Z_1^{-1} \left( Z_2 \xi_-(0) - \frac{1}{j\omega} \phi \right).$$

(58)

From this result the displacement at a distance $y$ from the boundary may be found by use of the transformation (19):

$$\xi_+(y) = X_1D_1(y)X_1^{-1}\xi_+(0).$$

(59)

Combining this with (55) and (58),

$$\xi_+(y) = X_1D_1(y)X_1^{-1}Z_1^{-1} \left( Z_1^T X_2 \xi_2(0) + \frac{1}{j\omega} \phi \right).$$

(60)

Summing the contributions (53), (54), and (60) we obtain

$$\xi(y, q''', q_z) = X_1D_1(y)\xi_1(y) + X_2D_2(y)\xi_2(y) + X_1D_1(y)X_1^{-1}Z_1^{-1} \left( Z_1^T X_2 \xi_2(0) + \frac{1}{j\omega} \phi \right).$$

(61)

This is the displacement corresponding to one particular $q''', q_z$. In order to obtain the total displacement we must compute the inverse Fourier transform with respect to $q'''$, and $q_z$ of (61), i.e.,

$$\hat{\xi}(y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi(y, q'', q_z) e^{-j(q''+q_z)}dq'dq_z.$$  

(62)

The evaluation of this double integral is in general quite complicated. However, if we confine our attention to the farfield region, approximate results of great interest are obtained. We have assumed already that the region of excitation is confined to the neighborhood of the origin and we attempt to compute the integral for points $(x, z)$ that are outside the region of excitation and at a large distance from the origin. After this approximate integration is carried out we shall see that $\hat{\xi}(y)$ consists of two different kinds of terms:

1) Terms that arise from points of stationary phase of the integrand; these terms represent volume waves.

2) Terms that arise from singularities in the integrand; these terms include surface waves.

We shall consider these two kinds of terms separately in the following sections.

**IV. Excitation of Volume Waves**

In (61) the components of $\xi_1(y)$ are zero for $y=0$. Their magnitudes increase with $y$ and approach an asymptotic value for $y>d$ where $d$ is the depth of the region of excitation (see Fig. 3). On the other hand, $\xi_2(y)$ has a finite nonzero value for $y=0$. Its components decrease in magnitude with increasing $y$ and become zero when $y\rightarrow d$. We shall evaluate the acoustic field at points of distance $y>d$ from the surface. For this reason we may neglect the second term in (61).

The integral (62) may then be written

$$\hat{\xi}(x, y, z) = \frac{1}{4\pi^2} \sum_{n=1}^{3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_n e^{-j(q''+q_z)} a_n dq'dq_z \tag{63}$$

where $a_n$ is a component of a column vector given by

$$\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix} = \kappa_1(\infty) + X_1^{-1}Z_1^{-1} \left[ Z_1^T X_2 \xi_2(0) + \frac{1}{j\omega} \phi \right].$$

(64)

We now seek the first term in the asymptotic expansion of the integral with respect to the distance from the origin. For this purpose we shall apply the well-known method of stationary phase. For large values of $(x^2+y^2+z^2)^{1/2}$ the phase

$$\phi = q''x + q_z z - n_{n}y$$

in general varies rapidly with $q''$ and $q_z$. The main contribution to the integral then comes from a small region around the so-called “points of stationary phase,” i.e., points satisfying

$$\frac{\partial}{\partial q''} \phi = 0; \quad \frac{\partial}{\partial q_z} \phi = 0.$$  

(66)

Let us now choose an $(q''', q_z)$ which is arbitrary within the range of values for which $n_1$, $n_2$, and $n_3$ are real. From (65) and (66), we then find the conditions that must be satisfied by $x$, $y$, and $z$ in order for $(q''', q_z)$ to be a point of stationary phase, i.e.,

$$x = \left( \frac{\partial n_m}{\partial q''} \right)_0 y; \quad z = \left( \frac{\partial n_m}{\partial q_z} \right)_0 y. \tag{67}$$

The direction in which the contribution from the neighborhood of $(q''', q_z)$ is radiated is defined in (67). This direction may also be expressed by the vector

$$\begin{bmatrix}
  \left( \frac{\partial n_m}{\partial q''} \right)_0 \\
  1 \\
  \left( \frac{\partial n_m}{\partial q_z} \right)_0
\end{bmatrix}, \tag{68}$$

a vector which may be shown to be colinear with the group velocity

$$\begin{bmatrix}
  \frac{\partial \omega}{\partial q''} \\
  \frac{\partial \omega}{\partial q''} \\
  \frac{\partial \omega}{\partial q_z}
\end{bmatrix} = v. \tag{69}$$

In the neighborhood of $(q''', q_z)$ the phase may now be written
\[ \phi = \phi_0 - \frac{y}{2} \left( \frac{\partial^2 n_m}{\partial q_x^2} \right)_0 (q_x - q_x^o)^2 - \frac{y}{2} \left( \frac{\partial^2 n_m}{\partial q_y^2} \right)_0 (q_y - q_y^o)^2 - y \left( \frac{\partial^2 n_m}{\partial q_x \partial q_y} \right)_0 (q_x - q_x^o)(q_y - q_y^o), \]  

where terms that are small of third-order have been neglected in the Taylor expansion.

In order to facilitate integration we introduce the new variables \( q_x' \) and \( q_y' \) defined by

\[ q_x - q_x^o = q_x' \cos \theta - q_x^o \sin \theta \]
\[ q_y - q_y^o = q_x^o \sin \theta + q_x' \cos \theta. \]  

By choosing a convenient value of the angle \( \theta \) we may always diagonalize the quadratic form (70) and obtain

\[ \phi = \phi_0 - \alpha_1 q_x'^2 - \alpha_2 q_y'^2. \]  

The transformation (71) is orthogonal and the determinants of the quadratic forms (70) and (72) are, therefore, equal. Hence

\[ \alpha_1 \alpha_2 = \frac{y^2}{4} \left( \frac{\partial^2 n_m}{\partial q_x^2} \right)_0 \left( \frac{\partial^2 n_m}{\partial q_y^2} \right)_0 - \left( \frac{\partial^2 n_m}{\partial q_x \partial q_y} \right)_0 \right). \]  

Here the quantity in the bracket is the Gaussian curvature of a wavenumber surface given by

\[ n - n(q_x, q_y) = 0. \]  

In general this equation describes three separate surfaces which are point symmetric with respect to the origin (see Fig. 4). Assuming that the coordinates of the point at which we seek the acoustic field satisfy (67), we observe that the main contribution to the integral comes from the \( m \)th wavenumber surface. As is evident from Fig. 4, the contributions from the other surfaces will be waves propagating in other directions. Denoting the radius of curvature of the \( m \)th surface at point \((q_x^o, q_y^o)\) by \( R_m \) we have

\[ \frac{1}{R_m^2} = \left( \frac{\partial^2 n_m}{\partial q_x^2} \right)_0 \left( \frac{\partial^2 n_m}{\partial q_y^2} \right)_0 - \left( \frac{\partial^2 n_m}{\partial q_x \partial q_y} \right)_0 \right). \]  

Combined with (73) this gives

\[ \left| \alpha_1 \alpha_2 \right|^{1/2} = \frac{1}{2} \frac{y}{R_m}. \]  

Assuming now that \( \xi_m \) and \( a_m \) are slowly varying functions of \( \hat{q} \) in the neighborhood of \( \hat{q} \), we may take them outside the integral signs and obtain

\[ \xi = \frac{1}{4\pi^3} \xi_m a_m e^{-i\xi_m q_x^o} \int_{-\infty}^{\infty} e^{-i\xi_m q_y^o} dq_y' \int_{-\infty}^{\infty} e^{-i\xi_m q_x^o} dq_x'. \]  

When the integrations are carried out and use is made of (76), we have

\[ \xi = \frac{1}{2\pi} \xi_m a_m e^{-i(\xi_m q_x^o + \xi_m q_y^o)} \frac{R_m}{y}. \]  

Here \( \psi \) is a phase factor that depends on the signs of \( \alpha_1 \) and \( \alpha_2 \). Introducing the angle \( \psi_m \) between the group velocity and the \( xz \) plane we may write (78) as

\[ \hat{k} = \frac{e^{i\psi}}{2\pi} \frac{R_m}{r \sin \psi_m} e^{-i\nu r} a_m \xi_m \]  

where \( \nu \) is the numerical value of the group velocity.

The mean energy density of the acoustic field is

\[ w = \frac{1}{2} s a^2 \xi^2 = \frac{s a^2}{8\pi^2} \left( \frac{R_m}{r \sin \psi_m} \right)^2 |a_m|^2 |\xi_m|^2 \xi_m \]  

and the mean density of power flow

\[ S = \nu w. \]

From (80) and (81) the total power converted into volume waves may be estimated.

The power radiated is

\[ P = \int \int \int \nu w \, dA, \]  

where \( dA \) is a differential area of the sphere of radius \( r \). We may now transform this expression to an integration over the wave-vector surface. If \( d\mathbf{A} \) is a differential area of a sphere of radius \( R_m \) we have

\[ dA = \left( \frac{r}{R_m} \right)^2 \, d\mathbf{A}. \]  

From (80), (82), and (83) we then obtain

\[ P = \frac{s a^2}{8\pi^2} \int \int \frac{1}{\sin \psi_m} \left| a_m \right|^2 |\xi_m|^2 \xi_m dA \]  

If the excitation has a narrow wave-vector spectrum centered around \((q_x^o, q_y^o)\) we may replace the wavenumber surface by its tangential plane at \((q_x^o, q_y^o)\). We then have

\[ dA = \frac{dq_x dq_y}{\sin \psi_m} \]  

and

\[ P = \frac{s a^2}{8\pi^2} \int \int \frac{1}{\sin \psi_m} |a_m|^2 |\xi_m|^2 \xi_m dq_x dq_y. \]

Another type of volume waves are the contribution to the integral from the branch-points of the integrand in (62). Waves of this type for an isotropic medium have been studied by Viktorov [2]. They will not be discussed here.
V. Excitation of Surface Waves

Surface waves arise for values of \( \vec{q} \) at which \( Z_1 \) is singular. For a given value of \( \omega \) we shall assume that \( Z_1 \) is singular when \( \vec{q} \) ends on a curve \( C \) in the \((q_x, q_z)\) plane (see Fig. 5).

Denoting the adjugate of \( Z_1 \) by \( Z_1^A \) we have for the inverse of \( Z_1 \)

\[
Z_1^{-1} = Z_1^A \frac{1}{|Z_1|}.
\]  
(86)

Let \((q_x^0, q_z^0)\) be a fixed point on \( C \). We shall suppose that \( q_x \) is kept constant and equal to \( q_x^0 \) while \( q_z \) varies in the neighborhood of \( q_z^0 \). Then (86) may be written

\[
Z_1^{-1} = Z_1^A \frac{1}{(q_x - q_x^0) \left( \frac{\partial}{\partial q_x} \right)_{q_z = q_z^0}}.
\]  
(87)

In (61) the third term will dominate for values of \((q_x, q_z)\) near the curve \( C \). For this reason we may write (61)

\[
\xi(y, q_x, q_z) = X_1 D_1(y) X_1^{-1} \xi_0 \frac{1}{(q_x - q_x^0) \left( \frac{\partial}{\partial q_x} \right)_{q_z = q_z^0}}
\]  
(88)

where

\[
\xi_0 = Z_1^A \left( Z_1^A X_1^T \xi_0(0) + \frac{1}{j\omega} \phi \right).
\]  
(89)

Inserting (88) into the integrand (62), we integrate first with respect to \( q_z \), and obtain

\[
I = - \frac{e^{-j\omega q_z}}{4\pi^2} \int_{-\infty}^{\infty} dq_z X_1 D_1(y) X_1^{-1} \xi_0
\]  
\[
\frac{1}{(q_x - q_x^0) \left( \frac{\partial}{\partial q_x} \right)_{q_z = q_z^0}} dq_z.
\]  
(90)

In evaluating this integral we choose a path of integration that circumscribes the pole at \( q_x^0 \) by a small semicircle. The semicircle is chosen to go above the real axis if \( q_x^0 > 0 \) and below if \( q_x^0 < 0 \). This choice insures that the contribution represents waves that propagate away from the region of excitation and may be regarded as a way of imposing the radiation condition.

Assuming that \( q_x^0 \) is the only pole of the integrand, the standard technique of contour integration gives the result

\[
I = - \frac{e^{-j\omega q_z}}{4\pi^2} \left( \frac{\partial}{\partial q_x} \right)_{q_z = q_z^0} \left( \frac{-2\pi j}{(q_x - q_x^0) \left( \frac{\partial}{\partial q_x} \right)_{q_z = q_z^0}} \right)_{q_x > 0}
\]  
(91)

This shows that the integral is nonzero only if

\[
q_x^0 > 0.
\]  
(92)

Let us regard as given the point \( \vec{r} = (x, z) \) at which we want to find the acoustic field. We are free to orient the coordinate system so that the \( q_x \) axis is parallel with \( \vec{r} \). In that case the inequality (92) may also be written

\[
\vec{q} \cdot \vec{\gamma} > 0.
\]  
(93)

Since the final result cannot depend on the orientation of the coordinate system we may use (93) which is invariant under coordinate transformations.

The amplitude \( \xi(y) \) of the resulting displacement at point \( \vec{r} \) is now found by completing the integration with respect to \( q_x \):

\[
\xi(y) = \frac{j}{2\pi} \int_{C_+} X_1 D_1(y) X_1^{-1} \xi_0 \left( \frac{\partial}{\partial q_x} \right)_{q_z = q_z^0} dq_x.
\]  
(94)

where the part of \( C \) for which (93) is satisfied is denoted by \( C_+ \). Furthermore \( \vec{q} = (q_x^0, q_z^0) \) has been replaced by \( \vec{q} \) which is now taken to be a point on \( C \). Since \( Z_1 \) is then singular, \( Z_1^A \) is of rank one and \( \xi_0 \) as defined by (89) is the null vector of \( Z_1 \).

When the exciting force is an ideal plane wave source of infinite extent in the \( x \) direction, its Fourier transform is a delta function in \( q_x \). The contribution to the integral then comes from a single point on the curve \( C_+ \), and an exact solution is easily found. In the general case the integral can only be solved approximately. An asymptotic solution valid for large distances from the origin can be found by the method of stationary phase. This problem has been dealt with by Buchwald [1]. We shall here give only the result of the integration. If we ask for the amplitude at a position \( \vec{r} \) from the origin, the contributions at this position come from those points on \( C \) where the outward normals are parallel with \( \vec{r} \). If \( \vec{q}^0 \) denotes such a point on \( C \), the displacement at \( \vec{r} \) is

\[
\hat{\xi}(y) = \frac{e^{j\pi/4}}{\sqrt{2\pi}} X_1 D_1(y) X_1^{-1} \xi_0 \left( \frac{R}{r} \right)^{1/2} \frac{e^{-j\vec{q}^0 \cdot \vec{r}}}{\left( \frac{\partial}{\partial \vec{p}} \right)_{\vec{q}^0} |Z_1|}.
\]  
(95)

Here \( R \) is the radius of curvature of \( C \) at the point \( \vec{q}^0 \) and \( p \) the wavenumber coordinate parallel to \( \vec{r} \) at \( \vec{q}^0 \). Thus \( \partial/\partial \vec{p} \cdot |Z_1| \) is the gradient in \((q_x, q_z)\) of \( |Z_1| \) at \( \vec{q}^0 \). If there is more than one point on \( C \) for which the external normal
points in the direction of \( \vec{r} \), the total displacement is a sum of terms, each given by (95).

When attention is confined to the displacement at the surface, (95) is simplified. Putting \( \gamma = 0 \) we obtain

\[
\hat{\xi}(0) = \frac{e^{i\omega t/4}}{\sqrt{2\pi}} \frac{R}{r} \frac{e^{-jR/2}}{\left| \frac{\partial}{\partial \vec{p}} | \vec{Z}_1 | \right|}.
\]

(96)

As one would expect for two-dimensional waves, the intensity decreases with distance proportionally to \( r^{-1/2} \).

For a medium which is elastically isotropic in the bounding plane, the curve \( C \) is a circle of radius \( R = \frac{2\pi}{\lambda} \)

(97)

where \( \lambda \) is the wavelength of the surface wave. For anisotropic media \( R \) may even be zero at particular points of \( C \).

The method of stationary phase in the form used above assumes the distance \( r \) so large that the exponential in (94) dominates the variation of the integrand. If \( R \) tends to zero the minimum distance \( r \) at which (96) is valid increases towards infinity. If \( R = 0 \) it is found that the dependence of \( \hat{\xi}(0) \) on distance will be given by the factor \( r^{-1/2} \), instead of \( r^{-1/2} \). Buchwald [1] has discussed these phenomena in detail.

In an earlier paper [9] expressions are given for the density of power flow in a surface wave. The \( x \) and \( z \) components of power flow are, respectively,

\[
S_x = \frac{\omega^2}{4j} \hat{\xi}_x^t(0) \left( \frac{\partial}{\partial q_x} \vec{Z}_1 \right) \hat{\xi}_x(0)
\]

(98)

\[
S_z = \frac{\omega^2}{4j} \hat{\xi}_z^t(0) \left( \frac{\partial}{\partial q_z} \vec{Z}_1 \right) \hat{\xi}_z(0).
\]

(99)

We assume that \( \xi(0) \) is a right null vector of \( N \). It then follows that \( \hat{\xi}_1(0) \) is the left null vector [9]. Making use of this we find that the power flow per unit length normal to the direction of flow is

\[
S = \left( S_x^2 + S_z^2 \right)^{1/2} = \frac{\omega^2}{8j} \hat{\xi}_0^t(0) \xi(0) (\partial \xi/\partial \vec{p})
\]

(100)

where \( z_1 \) is the eigenvalue of \( Z_1 \) that is zero on \( C \). When we insert for \( \hat{\xi}(0) \) from (96),

\[
S = \frac{\omega^2}{8j} \frac{R}{r} \frac{\xi(0)}{\left| \frac{\partial}{\partial \vec{p}} | \vec{Z}_1 | \right|}.
\]

(101)

For the adjugate of \( Z_1 \) we have the relation

\[
\xi_0^t Z_1 = z_1 z_2 \xi_0^t,
\]

where \( z_1 \) and \( z_2 \) are the two nonzero eigenvalues of \( Z_1 \). With \( t \) 's (101) may also be written

\[
S = \frac{\omega^2}{8j} \frac{R}{r} \frac{K / (\partial | \vec{Z}_1^* | / \partial \vec{p})}{\left| \frac{\partial}{\partial \vec{p}} | \vec{Z}_1 | \right|}.
\]

(102)

where

\[
K = \xi_0^t \left( \vec{Z}_1^* \vec{X}_1 \vec{K}_0(0) + \frac{1}{j\omega} \phi \right).
\]

(103)

The power flow from an ideal plane wave source is obtained from (102) by multiplication with \( (2\pi r/R) \), i.e.,

\[
S_{pw} = \frac{\omega^2}{4j} K / (\partial | \vec{Z}_1^* | / \partial \vec{p})
\]

(104)

Let us finally consider a particular example. Consider an applied force which is periodic in \( w \) with \( N \) periods and uniform in \( x \) over a width \( w \). Outside this domain the force is zero. Examples of an applied force of this kind are the comb-shaped transducers considered by Viktorov [2] and the interdigital electrode transducers first presented by White and Volmer [4]. The Fourier transformed force density which, according to (89), may be written as an equivalent surface force that also includes the effect of bulk forces is

\[
\frac{\sin \left( q_s \frac{2\pi N}{L} \right)}{q_s \left( q_s \frac{2\pi N}{L} \right) / \sin \left( \frac{q_s}{2} \right)}
\]

(105)

plus an equivalent term with \( -q_s \) substituted for \( q_s \). In (105) \( L = N \lambda \) is the total length in \( z \) direction of the applied force and \( A \) a vector which is independent of \( q_s \) and \( q_s \). We shall assume that this spectral distribution is so narrow that all quantities which represent material properties in (103) are the same in all directions close to the \( z \) direction, and for the sake of simplicity we assume that the group velocity is parallel to \( z \) when \( q_s = 0 \). Considered as a function of the angle \( \theta \) between the \( \vec{r} \) direction and the \( z \) direction, the power density will therefore depend only on the variation of the impressed force with \( \theta \). Introducing in (105)

\[
q_s = \frac{2\pi N}{L} \sin \theta, \quad q_s = \frac{2\pi N}{L} \cos \theta,
\]

(106)

we obtain

\[
S(\theta) = S(0) \left( L \frac{w^2}{2 \pi N^2} \right)^{1/2}
\]

(107)

\[
\sin^2 \left( \pi \frac{w \sin \theta}{L} \right) \sin^2 \left\{ 2\pi N (\cos \theta - 1) \right\}
\]

\[
\sin \left( \theta (\cos \theta - 1)^2 \right)
\]

Considered as a function of \( \theta \), the first zero in power density occurs when \( \theta = \theta_i \), where

\[
\theta_i = \min \left\{ \frac{\sin^{-1} \left( \frac{L}{N w} \right)}{\cos^{-1} \left( 1 - \frac{1}{2N} \right)} \right\}
\]

(108)

When \( w/L > 1/\sqrt{N} \) the first condition will dominate; otherwise, the latter will dominate.

The derivation of (95), (96), and (102) was based on a far-field approximation valid for large values of \( r \). However, the lower limit of \( r \) for which the approximation is valid was not given. For the particular example under consideration here, this lower limit is easily obtained. Let us assume, for
the sake of simplicity, that the wave-vector curve \( C \) is a circle. Requiring that the exponential function shall be, by a certain margin, the most rapidly varying factor in the integrand of (94), we find

\[
\frac{2r}{\lambda} \gg \delta^{-2},
\]

(109)

In the opposite extreme when \( 2r/\lambda \ll \delta^{-2} \), the plane wave approximation given by (104) is a fair approximation for the power density within the acoustic beam. It is worth noticing that in many reported laboratory experiments, the distance between transmitter and receiver has been in the intermediate range where neither approximation is applicable.

REFERENCES


Microsound Components, Circuits, and Applications

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Invited Paper

Abstract—Surface acoustic wave components have been realized which perform the functions of transduction, amplification, and coupling. Applications are suggested which make use of these components.

Exploratory work in connection with surface acoustic waveguides suggests the feasibility of acoustic analogs of conventional microwave transmission line (microsound) components on the surface of crystal and substrates. These microsound transmission lines, hybrids, and directional couplers interconnect microsound transducers, amplifiers, isolators, and phase shifters to form microsound circuits capable of autocorrelation, Fourier transformation, and cross correlation functions.

Compatible component configurations are proposed and evaluated which perform these basic functions. The anticipated difficulties with their realization are discussed and the current status of critical problems including the epitaxial growth of thin films and submicron etching procedures will be given. Several circuits capable of performing correlation functions are given.

I. INTRODUCTION

The exigencies of modern warfare require the acquisition and processing of immense quantities of data in very short periods of time. Perhaps the most demanding military problem is associated with antiballistic missile radar sensors in which the available information consisting of hundreds of thousands of microwave echoes with large bandwidths has to be processed in a few minutes. These sensors require delay-line systems with a bandwidth-delay time product perhaps as large as 10^4.

Similar problems also exist in the civilian sector. For example, if the cost of computer memory were reduced by a factor of 100, then large scale computer-aided education might become economic. The cost of computer memory has decreased with the size, bandwidth, and operating time of the individual components. Long wideband delay lines with fast access time may contribute to the cost reduction of circulating memory for special purpose computers. The interaction of acoustic and magnetoresistive films may lead to substantial cost reduction [1] of general computer memories. It may also be feasible to integrate acoustic and electronic circuits for more effective data processing.

The utility of sound for these applications in solids is related to the low propagation velocity and the excellent transmission characteristics of acoustic media. Sound travels five orders of magnitude more slowly than electromagnetic waves. It is possible to store a signal within one centimeter of crystal which ordinarily requires a 1-km long air-filled transmission line. The high Q of acoustic media permits delay times perhaps one hundred times that feasible with low-loss electromagnetic waveguide. Also, the wave-like nature of