Application of Microwave Concepts to the Theory of Acoustic Fields and Waves in Solids

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Invited Paper

Abstract—During the past 20 years the value of the microwave approach to electromagnetic field problems has been amply demonstrated. The purpose of this paper is to show the basic similarity of acoustic and electromagnetic field equations and to exploit this fact in applying microwave methods to acoustic resonator and waveguide problems. This is accomplished most directly and efficiently by using symbolic notation, rather than tensor subscripts, for the acoustic fields. The usefulness of this notation is illustrated by the problems of plane wave propagation and piezoelectric stiffening in an anisotropic medium, and by derivations of Poynting's and reciprocity theorems for a piezoelectric medium. Piezoelectric resonators are treated in detail from the point of view of normal mode expansions. A general network representation is obtained and is applied to the disk transducer, as an example. Normal mode theory of piezoelectric waveguides is briefly sketched and a perturbation theorem, which can be applied to both resonator and waveguide problems, is derived.

I. INTRODUCTION

The most fundamental concept in microwave electromagnetics is probably the reduction of field equations to ordinary differential equations by using normal mode expansions. From this, there has developed a number of secondary concepts [1], [2], circuit and scattering representations of resonators, discontinuities and junctions, coupled mode theory, tapered velocity coupling, and so on; and many perturbation, variational, relaxation, and, more recently, computer techniques have been applied to these concepts. With the growing interest in microwave acoustic wave applications, a need has arisen for solutions to resonator and guided wave problems similar to those encountered in microwave electromagnetics, and it is natural for engineers to look to microwave methods for solutions to these problems.

In general electromagnetic theory, Maxwell's equations are now almost always written in symbolic form. Vector quantities are indicated by arrows over the letter symbols or by boldface letters, rather than by giving the individual components. Components are used explicitly only when a specific problem is to be solved. This leads to greater clarity and to ease in manipulating the theory. By contrast, in acoustic (or elastic) wave theory, tensor or component notation is currently the most popular, although some recent publications [3]–[5] have used symbolic notation and pointed to its advantages.

In applying microwave methods to acoustic field problems, the advantages of symbolic notation seem to be particularly clear. First of all, it brings out the strong parallel between the electromagnetic and acoustic equations, thereby making it easier to transfer standard microwave concepts and techniques to acoustic problems. It also simplifies manipulation of the field equations and presentation of basic field theorems required for the normal mode theory.

II. ELECTROMAGNETIC AND ACOUSTIC FIELD EQUATIONS

The standard symbolic form for Maxwell's electromagnetic equations is

$$-\nabla \times E = \frac{\partial B}{\partial t}$$  \hspace{1cm} (1a)

$$\nabla \times H = \frac{\partial D}{\partial t} + J_s$$  \hspace{1cm} (1b)

For a lossless medium $J_s$ is the electrical source current density.

The basic acoustic field equations—Newton's equation and the strain-displacement relation—take the form [6], [7]

$$F_i + \frac{\partial}{\partial r_j} T_{ij} = \frac{\partial}{\partial t} \rho \frac{\partial u_i}{\partial t} \hspace{1cm} i, j = x, y, z$$  \hspace{1cm} (2a)

and

$$S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} \right)$$  \hspace{1cm} (2b)

in tensor notation. Here $F_i$ is the body force distribution, $\rho$ the mass density, $u_i$ the particle displacement, $r_i$ the position coordinate, and $T_{ij}, S_{ij}$ the second-order stress and strain tensors.

The acoustic equations are easily cast into a form parallel to (1a) and (1b) by using standard symbolism for the divergence of a second-order tensor,

$$\frac{\partial}{\partial r_j} T_{ij} = (\nabla \cdot T)_{ij}$$  \hspace{1cm} (3)

It will be seen in what follows that the symbolic equations are defined unambiguously by the use of boldface letters for all vector and tensor quantities. Standard letter symbols are used for all physical quantities and these specify the order of the tensor. With this information the operator symbols are clearly defined.
and indicating the symmetric part of the gradient of \( u \) by the symbol \( \nabla u \),

\[
\frac{1}{2} \left( \frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} \right) = (\nabla u)_{ij}.
\]

(4)

Introduction of the particle velocity field

\[
v = \frac{\partial}{\partial t} u
\]

(5)

and the particle momentum field

\[
p = \rho v
\]

(6)

then allows the acoustic equations to be written as

\[
\nabla \cdot T = \frac{\partial p}{\partial t} - F
\]

(7a)

\[
\nabla \times v = \frac{\partial S}{\partial t}.
\]

(7b)

These already bear a strong resemblance to the electromagnetic equations (1a) and (1b), and the analogy will become even clearer when the properties of the operators \( \nabla \cdot \) and \( \nabla \times \) are examined in more detail.

Obviously a field symbolism is useful only if it can be easily translated into component form for the analysis of specific problems. In electromagnetism it is common to display the field components in a matrix representation as single column matrices

\[
E = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \quad H = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}, \quad \text{etc.}
\]

(8)

and the curl operator is represented as a matrix-differential operator

\[
[\nabla \times] = \begin{bmatrix} 0 & -\partial/\partial y & \partial/\partial z \\ -\partial/\partial x & 0 & -\partial/\partial z \\ -\partial/\partial y & \partial/\partial x & 0 \end{bmatrix} = \begin{bmatrix} \nabla \times \end{bmatrix}.
\]

(9)

Here the tilde (\( \sim \)) indicates the transpose matrix. The operators in (1a) and (1b) are therefore transposes of each other.

It is customary to obtain matrix representations for the strain and stress fields by using the reduced (or Voigt) subscript notation [6]. The strain field is, in general, a symmetric second-order tensor

\[
S_{ij} = S_{ji}, \quad i, j = x, y, z
\]

and therefore has only six independent components. In reduced subscript notation, these are designated

\[
S_1 = S_{xx}, \quad S_4 = 2S_{yx},
\]

\[
S_2 = S_{yy}, \quad S_5 = 2S_{zx},
\]

\[
S_3 = S_{zz}, \quad S_6 = 2S_{xy}
\]

and the strain is therefore represented as a six-row column matrix

\[
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix}
\]

(10)

By contrast, the stress tensor is symmetric only when the medium does not possess a permanent electric or magnetic moment [8], [9]. Only in this case can the reduced subscript notation

\[
T_1 = T_{xx}, \quad T_4 = T_{yx},
\]

\[
T_2 = T_{yy}, \quad T_5 = T_{zx},
\]

\[
T_3 = T_{zz}, \quad T_6 = T_{xy}
\]

be used with strict accuracy and the stress represented exactly by a six-row column vector

\[
T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix}
\]

(11)

In this matrix representation it is easily determined from (3) and (4) that the operators in the acoustic field equations have representations given by

\[
[\nabla_\times] = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \end{bmatrix} = [\nabla \times].
\]

(12)

That is, the acoustic field operators are transposes of each other, just as the electromagnetic operators were. A further point of similarity lies in the strong parallelism between the electromagnetic and acoustic operator identities used in Sections V and VI.

III. CONSTITUTIVE RELATIONS

Both the electromagnetic and acoustic field equations involve four different field quantities, \( E, B, H, D \) in the electromagnetic case and \( T, p, v, S \) in the acoustic case. These fields are related to each other through the properties of the medium, which have not yet been introduced. In each system, two field quantities are taken as basic and the others are

\[\text{It is, however, common in the current literature for the stress tensor in polar media, such as lithium niobate, to be treated as symmetric.}\]
expressed in terms of these by means of constitutive relations defining the medium. For the electromagnetic field, it is customary to take \( E \) and \( H \) as the basic variables. Comparison of (1a) and (1b) with (7a) and (7b) shows that the analogous choice for the acoustic field is \( T \) and \( v \).

The treatment given here will not include magnetostrictive interaction between the electromagnetic and acoustic fields. Also, only small signal theory will be considered. The constitutive relations will therefore be linear. In the anisotropic case,\(^4\) these then have the general form

\[
B = \mathbf{u} \cdot H \\
D = \varepsilon^p \cdot E + \mathbf{d} \cdot T \\
p = \rho v \\
S = \mathbf{d} \cdot E + s^E : T. \tag{13d}
\]

Here \( \mathbf{u} \) and \( \varepsilon \) are the second-order permeability and permitivity tensors, and the single dot indicates the ordinary scalar product

\[
\mathbf{u} \cdot H = \sum_i \mu_i H_i \quad i, j = x, y, z.
\]

The superscript \( T \) on \( \varepsilon \) shows that this is the permitivity at zero (or constant) stress \( T \).

In (13b) the quantity \( \mathbf{d} \) is the third-order piezoelectric strain tensor and the double dot indicates the usual double scalar product [5]. This is defined in terms of either full or reduced subscripts as

\[
\mathbf{d} : T = \sum_{i,j,k} \mathbf{d}_{ijk} T_{jk} = \sum_{i,j} \mathbf{d}_{ij} T_{j} \tag{14}
\]

The latter form is preferable here because it allows the piezoelectric strain tensor to be represented by a 3-column, 6-row matrix [6]. In (13d) the ordinary scalar product \( \mathbf{d} \cdot E \) is represented in either full or reduced subscripts as

\[
\mathbf{d} \cdot E = \sum_k \mathbf{d}_{k} E_k = \sum_{i,j} \mathbf{d}_{ij} E_i \tag{15}
\]

Since it can be shown that [6]

\[
\mathbf{d}_{ij} = \mathbf{d}_{ji},
\]

it is not necessary to have a separate symbol for the tensor in (15). The matrix representation required is defined by the dot operation used.

The symbol \( s^E \) in (13d) is the fourth-order elastic compliance tensor. In reduced subscript notation this is represented by a 6\( \times \)6 matrix which can be shown to be symmetric [6],

\[
s_{ij}^E = s_{ji}^E \tag{17}
\]

and the double dot indicates the double scalar product

\[
s^E : T = \sum_{i,j} s_{ij}^E T_{ij},
\]

The superscript \( E \) on \( s \) shows that this is the compliance at zero (or constant) electric field \( E \).

For some purposes it is desirable to use \( S \) rather than \( T \) as one of the independent acoustic variables. The piezoelectric interaction equations for this case are easily obtained by first considering the inverse of the compliance tensor. This is the stiffness tensor

\[
c^E, \tag{18}
\]

which has a matrix representation \( c_{ij}^E = c_{ji}^E \) such that

\[
c_{ij}^E \delta_{jk} = \delta_{ik}. \tag{19}
\]

If the double scalar product of (13d) is taken with \( c^E \) and use is made of (18), the result can be rearranged as

\[
T = - c^E : \mathbf{d} \cdot E + c^E : S = - e \cdot E + c^E : S. \tag{19}
\]

Here,

\[
e = c^E : \mathbf{d},
\]

the piezoelectric stress tensor, has the matrix representation

\[
e_{ij} = \sum_j c_{ij}^E d_{jk} = \sum_j d_{ij} c_{ji}^E = e_{ij}, \tag{20}
\]

from (16) and the symmetry of \( c^E \). Substitution of (19) for \( T \) in (13b) then gives

\[
D = \varepsilon^p \cdot E + \varepsilon : S, \tag{21}
\]

with

\[
\varepsilon^p = \varepsilon^p - \mathbf{d} : c^E : \mathbf{d}. \tag{22}
\]

IV. PLANE WAVE PROPAGATION IN ANISOTROPIC MEDIA

\textbf{Christoffel Equation}

In electromagnetism the electric and magnetic wave equations are obtained by eliminating \( H \) or \( E \) from the symbolic field equations (1a) and (1b) and the constitutive relations. Similarly, acoustic wave equations are obtained from (7a) and (7b) by eliminating either \( T \) or \( v \). Since \( v \) has fewer components than \( T \), the velocity equation is more convenient. The double scalar product of (7b) is taken with \( c^E \). Use of the piezoelectric stress equation (19) then gives

\[
c^E : \nabla \nu = c^E : \frac{\partial S}{\partial \nu} = e : \frac{\partial E}{\partial t} + \frac{\partial T}{\partial t}. \tag{23}
\]

From the time derivative of (7a)

\[
\frac{\partial T}{\partial t} = \rho \frac{\partial v}{\partial t} - \frac{\partial F}{\partial t}. \tag{24}
\]
Elimination of $\partial T/\partial t$ gives the velocity wave equation with piezoelectric and body force driving terms

$$\nabla \cdot e : \nabla \nu = \rho \frac{\partial \nu}{\partial t} + \nabla \cdot e \cdot \frac{\partial F}{\partial t} = \frac{\partial F}{\partial t} .$$

(25)

By using (12) and the matrix representation of the stiffness tensor, the left-hand side of (25) can be represented as a $3 \times 3$ matrix-differential operator. For a nonpiezoelectric medium with no body force sources ($e=F=0$) and plane wave solutions of the form $e^{i(\omega t-k \cdot r)}$,

$$\frac{\partial}{\partial t} = -jk,$$

$$\frac{\partial}{\partial t} = j\omega.$$

The wave equation, in matrix representation, then becomes

$$\sum \Gamma_{ij} v_j = -\omega^2 v_i = 0,$$

(26)

with

$$\begin{bmatrix}
\Gamma_{ij} \\
\end{bmatrix} =
\begin{bmatrix}
k_x & 0 & 0 & k_x & k_y \\
0 & k_y & 0 & k_x & 0 \\
0 & 0 & k_y & k_x & 0
\end{bmatrix}
$$

called the Christoffel equation in the elasticity literature. This presentation gives a simple method for obtaining the Christoffel equation from the stiffness matrix. Plane wave propagation velocities and field distributions can then be calculated.

**Piezoelectric Stiffening**

In a piezoelectric medium, an electrical field is present even when there are no electrical sources, and the plane wave analysis must be modified to take account of this fact. For waves traveling much slower than the electromagnetic velocity, (1a) and (1b) can be approximated by the quasi-electrostatic equations

$$\nabla \times E = 0$$

(27a)

and

$$\nabla \cdot D = 0.$$  

(27b)

For a plane wave solution,

$$\nabla = -jk;$$

and (27a) is equivalent to

$$k \times E = 0,$$

or

$$E = C k.$$

which is the source of $E$, is distributed as shown in the figure and can only produce a field along $k$. 

Fig. 1. Electric field distribution in a piezoelectric plane wave. $Z_k$ is the acoustic wave impedance.

The constant $C$ must be proportional to the particle velocity, because the electric and acoustic fields are coupled by means of the piezoelectric interaction. It is evaluated by substituting (21) into

$$k \cdot D = 0,$$

(28)

which is obtained from (27b), and using the strain–velocity relation

$$S = \nabla \nu / j\omega,$$

given by (7b). The piezoelectrically induced field is then

$$E = -k (k \cdot e : \nabla \nu / j\omega) / k \cdot e^2 \cdot k .$$

(29)

Even in media with anisotropic permittivity the induced field is always parallel to $k$. The physical reason for this is illustrated in Fig. 1. In general, the piezoelectrically induced electrical displacement

$$e \cdot S$$

has an arbitrary direction. However, the fields vary only along $k$. Consequently, the piezoelectrically induced charge density

$$\nabla \cdot e \cdot S,$$

which is the source of $E$, is distributed as shown in the figure and can only produce a field along $k$. 

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After the piezoelectrically induced electric field has been evaluated, it can be substituted back into (25), with \( F = 0 \), to determine the reaction upon the acoustic field. The result is

\[
\nabla \cdot \left( c^E + \frac{(e \cdot k)(k \cdot e)}{k \cdot e^S \cdot k} \right) \nabla \nu = -\rho_0 \omega \nu.
\]

This shows that the effect of piezoelectricity is to modify or stiffen the elastic constants. The stiffened elastic tensor,

\[
c^k \cdot D = c^E + \frac{(e \cdot k)(k \cdot e)}{k \cdot e^S \cdot k}
\]

where the superscript \( k \cdot D \) follows from (28), can then be used directly in the Christoffel equation.

V. POYNTING'S THEOREMS

In electromagnetism, the real and complex Poynting's theorems [10] are useful tools for the study of power flow and energy balance relations. Corresponding relations are just as easily derived for the electromagnetic and acoustic fields in a piezoelectric medium.

Only the complex theorem, which applies to fields varying as \( e^{\omega t} \), will be derived here. The standard electromagnetic derivation is to add the scalar product of (1b) with \( E^* \) to the scalar product of \( H \) with the complex conjugate of (1a). Use of the identity

\[
\nabla \cdot (E^* \times H) = H \cdot \nabla \times E^* - E^* \cdot \nabla \times H
\]

then gives

\[
-\nabla \cdot E^* \times H = j\omega (E^* \cdot D - H \cdot B^*) + E^* \cdot J^*.
\]

For the acoustic equations, the double scalar product of \( T^* \) with (7b) is added to the scalar product of \( \nu \) with the complex conjugate of (7a). The acoustic analog [5] of the electromagnetic identity (31),

\[
\nabla \cdot (\nu \cdot T^*) = \nu \cdot (\nabla \cdot T^*) + T^* : \nabla \nu,
\]

is applied to the sum, giving

\[
\nabla \cdot (\nu \cdot T^*) = j\omega (\nu \cdot p^* + T^* : S) - \nu \cdot F^*.
\]

If (32) is added to (34) and integrated over a volume \( V \) bounded by a surface \( S \), with outward normal \( n \), application of the divergence theorem leads to the complex Poynting's theorem

\[
\int_S (\nu \cdot T^* - E^* \times H) \cdot n \, dS = j\omega \int_V (E^* \cdot D + T^* : S - H \cdot B^* - \nu \cdot p^*) \, dV
\]

\[
= j\omega \int_V (E^* \cdot D + T^* : S - \nu \cdot F^*) \, dV.
\]

Material parameters are introduced by means of the constitutive relations (13a)-(13d). Under the surface integral,

\[
P_e = E^* \times H
\]

is the electrical complex Poynting's vector. The identification of

\[
P_a = -\nu \cdot T^*
\]

with the acoustic complex Poynting's vector is illustrated in Fig. 2.

VI. RECIPROCITY RELATIONS

Two other important analytical tools in electromagnetism are the real (or Lorentz) and complex reciprocity relations [10], [11]. Their piezoelectric counterparts are equally important. As in Section V, only the complex relation will be considered.

The derivation is performed most efficiently by combining the field equations (1a), (1b) and (7a), (7b) with the constitutive relations in a single matrix-type equation

\[
\begin{bmatrix}
\rho & 0 & 0 & 0 \\
0 & s & 0 & d \\
0 & d & 0 & \nu \\
0 & d & 0 & \nu^* \\
\end{bmatrix}
\begin{bmatrix}
\nabla \cdot T \\
\nabla \times H \\
\n\nabla \times \nu \\
\n\nabla \cdot \nu \\
\end{bmatrix}
= \begin{bmatrix}
v \\
T \\
\nu \\
J \\
\end{bmatrix}
\]

where multiplication is carried out according to the usual matrix rules. Two field solutions are assumed: 1) \( v_1, T_1, H_1, E_1 \), driven by sources \( F_1, J_1 \), and 2) \( v_2, T_2, H_2, E_2 \), driven by sources \( F_2, J_2 \). Solution 1) is written into (36) and a matrix-type scalar product is taken with the complex conjugate of solution 2). This gives

\[
v_1^* \cdot (\nabla \cdot T_1) + T_1^* : \nabla v_1 - H_1^* \cdot \nabla \times E_1 + E_1^* \cdot \nabla \times H_1
\]

\[
= [v_2^*T_2^*H_2^*E_2^*]
\begin{bmatrix}
\rho & 0 & 0 & 0 \\
0 & s & 0 & d \\
0 & d & 0 & \nu \\
0 & d & 0 & \nu^* \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
T_1 \\
H_1 \\
E_1 \\
\end{bmatrix}
\]

\[-v_2^* \cdot F_1 + E_2^* \cdot J_1.
\]

A second equation, obtained by complex conjugation and interchange of subscripts, is added to this. Since the permeability and permittivity tensors \( \nu, \epsilon \) are symmetric [6], and so are \( \nu \) and \( d \), from (16) and (17), the array of constitutive parameters in (36) is symmetric. By using this fact and the

\footnote{The case of a magnetic or conducting medium which interacts with a dc magnetic field will not be considered.}
identities (31), (33), the sum of the two equations is reduced to the complex reciprocity relation
\[
\nabla \cdot (-v_2^* \cdot T_1 - v_1 \cdot T_2^* + \nabla \cdot (E_2^* \times H_1 + E_1 \times H_2^*)
\]
\[
= -\frac{\partial}{\partial t} \left[ \begin{array}{c} v_2^* T_2^* H_2^* E_2^* \\ 0 : s : d : 0 \\ 0 : 0 : 0 : d \\ 0 \cdot d : 0 : \varepsilon \cdot \tau \\ E_1 \end{array} \right] + (v_2^* \cdot F_1 + v_1 \cdot F_2^*) - (E_1^* \cdot J_1 + E_2^* \cdot J_2^*). \tag{37}
\]

In the quasi-electrostatic approximation,
\[
\nabla \times E = 0 \\
E = -\nabla \phi,
\]
this relation is considerably simplified. From (31) and (1b) the electromagnetic divergence terms transform into
\[
\nabla \cdot (E_2^* \times H_1) = \nabla \phi_2^* \cdot \nabla \times H_2 = -\nabla \phi_2^* \cdot \frac{\partial D_1}{\partial t} + \nabla \phi_2^* \cdot J_1.
\]

Use of the identity
\[
\nabla \cdot (\phi_2^* D_1) = \phi_2^* \nabla \cdot D_1 + \nabla \phi_2^* \cdot D_1 \tag{38}
\]
and the equation of continuity
\[
\nabla \cdot \frac{\partial D}{\partial t} = \frac{\partial \rho_e}{\partial t}
\]
transforms this further into
\[
\nabla \cdot (E_2^* \times H_1) = \nabla \left( \phi_2^* \frac{\partial D_1}{\partial t} \right) - \phi_2^* \frac{\partial \rho_e}{\partial t} + \nabla \phi_2^* \cdot J_1.
\]

From this, the quasi-electrostatic form of the reciprocity relation is
\[
\nabla \cdot \left( -v_2^* \cdot T_1 - v_1 \cdot T_2^* + \phi_2^* \frac{\partial D_1}{\partial t} + \phi_1 \frac{\partial D_2^*}{\partial t} \right)
\]
\[
= -\frac{\partial}{\partial t} \left[ \begin{array}{c} v_2^* T_2^* - \nabla \phi_2^* \\ 0 : s : d : 0 \\ 0 \cdot d : 0 : \varepsilon \cdot \tau \\ -\nabla \phi_1 \end{array} \right] + v_2^* \cdot F_1 + v_1 \cdot F_2^* + \phi_2^* \frac{\partial \rho_e}{\partial t} + \phi_1 \frac{\partial \rho_e^*}{\partial t}. \tag{39}
\]

Note that the electric current sources have now been replaced by electric charge sources. This important relation provides the basis for the normal mode theory of piezoelectric resonators and waveguides.

VII. PIEZOELECTRIC RESONATORS

Except for notational details, the approach used here will be the same as that developed by Lewis [12] and Lloyd [13]. The configuration, shown in Fig. 3, is an arbitrarily shaped

\[
\int_S (-v_n^* \cdot T_m - v_m \cdot T_n^* + \phi_n^* [j \omega_m D_m] + \phi_m [j \omega_n D_n]^*) \cdot n \, dS
\]
\[
= -j (\omega_m - \omega_n) \int_V [v_n^* T_m^* - \nabla \phi_n^*] \begin{bmatrix} \rho & 0 & 0 \\ 0 : s : d : 0 \\ 0 \cdot d : 0 : \varepsilon \cdot \tau \end{bmatrix} \begin{bmatrix} v_m \\ T_m \end{bmatrix} dV. \tag{41}
\]
TABLE I
GENERAL ELECTRICAL AND ACOUSTIC BOUNDARY CONDITIONS
FOR THE ELECTRODED REGIONS \( S_p \)

| \( \phi_p \) = constant |
| \( \int_{S_p} \frac{\partial}{\partial t} D \cdot n \, dS = - \frac{\partial}{\partial t} Q_n = - I_p \) |
| \( (T \cdot n)_p = - \rho_n \frac{\partial v}{\partial t} \) |

The resonator boundary \( S \) in Fig. 3 can be divided into \( N \) electroded regions \( S_p \) and an unelectroded region \( S_u \). From the perfect conductivity of the electrodes the potential function \( \phi \) must be a constant over each electrode; and, since the electrode material is assumed to have no stiffness, the traction force \( \tau = T \cdot n \) on \( S_p \) [Fig. 3(b)] is determined completely by the mass loading. Boundary conditions on the electroded regions are therefore as given in Table I. For a lossless resonator the electrical boundary conditions on the electrodes must be chosen so that there is no average power flow out of the terminals. This permits connection of an arbitrary reactive circuit to each terminal. However, only the simplest terminations, short-circuit (\( \phi_p = 0 \)) and open-circuit (\( I_p = 0 \)), will be considered here.

On the unelectroded boundary region \( S_u \), any set of lossless boundary conditions, including mass and capacitance loading, may be used. In this presentation only the boundary conditions in Table II will be considered. These may be uniform over the entire region. Alternatively, either discrete or continuous variations with position on \( S_u \) may be permitted.

Substitution of the boundary conditions from Tables I and II into (41) gives

\[
j(\omega_m - \omega_n) \left\{ \rho_s \sum_p n_p \int_{S_p} v_m^* \cdot v_m \, dS \right. \\
+ \int_V \left[ \begin{array}{c} \rho \\ 0 \end{array} \right] \cdot \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{D} \end{array} \right] \cdot \left[ \begin{array}{c} \mathbf{v}_m \\ \mathbf{T}_m \end{array} \right] \, dV \right. \\
\left. = \sum_p (\phi_p)_m (I_p)_n + (\phi_p)_m (I_p)_n^* \right. \tag{42}
\]

With short-circuit or open-circuit terminations on the electrodes, this becomes

\[
j(\omega_m - \omega_n) (\Psi_m^*, \Psi_m) = 0,
\]

where

\[
(\Psi_m^*, \Psi_m) \equiv \rho_s \sum_{p=0}^{N} \int_{S_p} v_m^* \cdot v_m \, dS
\]

\[
+ \int_V \left[ \begin{array}{c} \rho \\ 0 \end{array} \right] \cdot \left[ \begin{array}{c} \mathbf{E} \\ \mathbf{D} \end{array} \right] \cdot \left[ \begin{array}{c} \mathbf{v}_m \\ \mathbf{T}_m \end{array} \right] \, dV \right. \tag{43}
\]

Normal Mode Expansion of Forced Oscillation Fields

The general forced piezoelectric resonator problem is illustrated in Fig. 4. In addition to the electrical excitation considered previously [12], [13], provision is made for acoustic excitation by means of acoustic waveguides which contact the resonator surface in regions \( S_p (N < p \leq N') \). It will be assumed that there are no volume sources, \( F \) or \( P_e \). These are easily included, if needed.

The analytical procedure is to take solution 1) in the reciprocity relation (39) to be a forced solution

\[
\mathbf{v}_1 = \mathbf{v}(x, y, z, t), \text{ etc.,}
\]

and solution 2) to be a free normal mode

\[
\mathbf{v}_n = e^{i \omega_n t} \mathbf{v}_n(x, y, z).
\]

Boundary conditions for the normal modes have already been defined. The forced solution will satisfy the boundary conditions of Table I on all electrodes (0 \( \leq p \leq N \)) and the boundary conditions of Table II on all parts of the boundary exclusive of the electrodes and the acoustic ports. At the acoustic ports, the electrical boundary conditions in Table II are applied, but the acoustic boundary conditions are left unspecified. Similarly, the currents and potentials at the electrodes are unspecified, except that the zero electrode is defined as ground (\( \phi_0 = 0 \)). After integration over the resonator volume \( V \) and substitution of boundary conditions the
reciprocity relation can be converted to the form

\[
\left( \frac{\partial}{\partial t} - j\omega_n \right) (\Psi_n^* , \Psi) = \sum_{p=1}^{N} \left\{ (\phi_p)n^* I_p + (I_p)n^* \right\} + \sum_{p=N+1}^{N'} \int_{S_p} \{ v_n^* \cdot T + v \cdot T_n^* \} \cdot n \, dS
\]

(45)

where \((\Psi_n^* , \Psi)\) is defined according to (43).

At this point the discussion will be restricted to normal modes which satisfy electric open-circuit boundary conditions at the electrodes and rigid (or acoustic open-circuit) conditions at the acoustic ports. The relation above then reduces, for the case of a steady state solution at frequency \(\omega\), to

\[
j(\omega - \omega_0(n)) (\Psi_n^* , \Psi) = \sum_{p=1}^{N} (\phi_p)n^* I_p \]

(46)

\[
\quad + \sum_{p=N+1}^{N'} \int_{S_p} v_p^* \cdot T_n^* \cdot n \, dS,
\]

where the subscript 0(n) denotes the open-circuit normal modes. This shows that the forced solution can be specified by giving the driving currents \(I_p\) at the electrodes and the driving velocity distributions \(v_p\) at the acoustic ports.

In the normal mode theory of electromagnetic resonators it is now well known that static field solutions must often be included in expanding forced oscillation fields [14]. This requirement also occurs in piezoelectric resonator problems. The general expansion for a solution at frequency \(\omega\) is therefore written

\[
\Psi = \Psi_t + \sum_n a_n \Psi_0(n) , \quad (47)
\]

where

\[
\Psi = \begin{bmatrix} v \\ T \\ -\nabla \phi \end{bmatrix}
\]

and

\[
\Psi_t = \begin{bmatrix} 0 \\ T_t \\ -\nabla \phi_t \end{bmatrix}
\]

is the solution \(v = 0, u, T, -\nabla \phi\) to the static \((\omega \to 0)\) boundary value problem with the same boundary conditions as the forced solution \(\Psi\), but with

\[
(Q_p)_s = \frac{I_p}{j\omega} \quad \text{on} \quad S_p(1 \leq p \leq N) \quad (48a)
\]

\[
(a_p)_s = \frac{v_p}{j\omega} \quad \text{on} \quad S_p(N < p \leq N'). \quad (48b)
\]

From (47) and (44),

\[
(\Psi_0^* , \Psi) = (\Psi_0^* , \Psi) + a_n (\Psi_0^* , \Psi_0(n)).
\]

The first term on the right-hand side can be eliminated by applying the reciprocity relation to the \(n\)th mode field and the static field. This is easily accomplished by substituting \(\Psi\) for \(\Psi\) in (45). After taking into account the boundary conditions for the open-circuit modes, and the fact that \(\omega, (I_p)_s, v_s, 0\), this gives

\[
(\Psi_0^* , \Psi) = 0.
\]

Consequently, (46) defines the mode expansion coefficients in terms of the driving currents and velocities,

\[
a_n = \frac{\sum_{p=1}^{N} (\phi_p)n_I_p + \sum_{p=N+1}^{N'} \int_{S_p} v_p^* \cdot T_0(n) \cdot n \, dS}{j(\omega - \omega_0(n)) (\Psi_0(n), \Psi_0(n)).}
\]

(49)

The normal mode fields can always be chosen so that \(\phi_0(n)\) and \(T_0(n)\) are pure real. For convenience this has been done and the complex conjugates removed from the right-hand side of (46).

From these expansion coefficients, the reaction of the forced oscillation back on the driving sources can be evaluated; and, from this, the impedance or scattering matrix representation of the resonator may be calculated. To illustrate the procedure, two examples will be considered, an arbitrary resonator with \(N\) electrical ports and a thin disc transducer with two acoustic ports.

**N-Electrode Resonator**

In this case there are no acoustic ports and the acoustic driving terms are omitted from (49). From the normal mode expansion (47), the forced potential at the \(q\)th electrode is

\[
\phi_q = (\phi_q)_q + \sum_n a_n (\phi_0(n))_q.
\]

Substitution for the normal mode amplitudes gives

\[
\phi_q = (\phi_q)_q + \sum_{p=1}^{N} \sum_n \frac{(\phi_0(n))_q (\phi_0(n))_p I_p}{j(\omega - \omega_0(n)) (\Psi_0^*(n), \Psi_0(n))} . \quad (50)
\]

From each normal mode solution, another solution can always be obtained by taking the complex conjugate. That is, the modes must occur in complex conjugate pairs,

\[
\Psi_0(n)e^{j\omega_0(n)t}
\]

and

\[
\Psi_0^*(n)e^{-j\omega_0(n)t},
\]

one with a positive frequency and the other with a negative frequency. It is convenient to designate each mode having a positive frequency by a positive mode index \(n\) and its partner by a negative index \(-|n|\); namely,

\[
\omega_0(|n|) > 0 \quad \omega_0(-|n|) = -\omega_0(|n|).
\]
Since the normal mode potential fields $\phi_0(n)$ are chosen to be real, modes can be combined in pairs in (50), giving

$$\phi_q = (\phi_q)_2 + \sum_p \sum_{n1} \frac{\omega}{j(\omega^2 - \omega_0^2(n1))} F_{qp}^{(n1)} I_p,$$  \hspace{1cm} (51)

with

$$F_{qp}^{(n1)} = \frac{2(\phi_0(n))_2(\phi_0(n))_p}{\langle \Psi_0^*(n), \Psi_0(n) \rangle}.$$  

For the static boundary value problem, the potential-charge relations are defined as

$$(\phi_q)_2 = \sum_p F_{qp}^{(*)} (Q_q)_p.$$  \hspace{1cm} (48a)

Substitution of this into (51) and use of the boundary condition (48a) then gives

$$\phi_q = \frac{1}{j\omega} \sum_p \left\{ F_{qp}^{(*)} - \sum_{n1} \frac{\omega^2}{\omega_0^2(n1) - \omega^2} F_{qp}^{(n1)} \right\} I_p.$$  \hspace{1cm} (52)

This is an impedance matrix relation, with impedances

$$Z_{qp} = \frac{1}{j\omega} \left\{ F_{qp}^{(*)} - \sum_{n1} \frac{\omega^2}{\omega_0^2(n1) - \omega^2} F_{qp}^{(n1)} \right\}$$  \hspace{1cm} (53)

which have the electrical circuit representations shown in Fig. 5.

It appears that the coefficients $F_{qp}^{(n1)}$, defined above, differ from those given by Lloyd [13]. There is, however, a difference in the mode normalization factors. From the constitutive relations (13a)-(13d) and the definition (43),

$$(\Psi_0(n), \Psi_0(n)) = \sum_{n2} \int_{S_p} \Psi_0^*(n2) \Psi_0(n2) dS + \int_V \nabla \Psi_0^{(*)} \cdot \nabla \Psi_0(n2) \cdot D_0(n2) dV \}.$$  \hspace{1cm} (43)

By applying the quasi-electrostatic form of Poynting's theorem (35) to the normal mode fields and taking note of boundary conditions, the relation

$$\int_V (\nabla \Psi_0^{(*)} \cdot \nabla \Psi_0(n2) \cdot D_0(n2) dV = \int_V \Psi_0(n2) \cdot \Psi_0^*(n2) dV + \rho_s \sum_{n1} \int_{S_p} \Psi_0^*(n1) \Psi_0(n1) dS$$

is established. From this,

$$(\Psi_0(n), \Psi_0(n)) = 2 \rho_s \sum_{n1} \int_{S_p} \Psi_0^*(n1) \Psi_0(n1) dS + \int_V \Psi_0^*(n2) \Psi_0(n2) dV = 2 \omega_0(n) V \left( \Psi_0(n1), \Psi_0(n1) \right),$$

where $V(\Psi_0(n1), \Psi_0(n1))$ is Lloyd's normalization factor.

**Thin Disk Transducer**

Consider a thin disk shear transducer using a cubic piezoelectric material, such as bismuth germanium oxide [15], oriented as shown in Fig. 6. A uniform plane wave model will be assumed. Solution of the plane wave problem, following Section IV, shows that pure shear waves with vertical polarization propagate along the $x'$ direction. The traction forces and velocities at the acoustic ports are therefore directed as shown in the figure.

In this problem the mode expansion coefficients are, from (49),

$$a_n = \frac{(\phi_0(n))_1 I_1 + \sum_p v_p (\tau_0(n))_p}{j(\omega - \omega_0(n)) (\Psi_0^*(n), \Psi_0(n))}. \hspace{1cm} (54)$$

The normal mode expansions for the forced potential and traction forces are, from (47),

$$\phi_1 = (\phi_1)_1 + \sum_n a_n (\phi_0(n))_1$$

$$\tau_q A = (\tau_q)_q A + \sum_n a_n (\tau_0(n))_2 A \hspace{1cm} q = 2, 3.$$  \hspace{1cm} (55)

As in the purely electrical problem just treated, the mode expansion coefficients are substituted from (53) and complex mode pairs are combined. This gives

$$\phi_1 = (\phi_1)_1 + \sum_{n1} \rho \int_{S_p} \frac{\omega}{j(\omega^2 - \omega_0^2(n1))} \{ F_{11}^{(n1)} I_1 + \sum_p F_{1p}^{(n1)} I_p \}$$

$$\tau_q A = (\tau_q)_q + \sum_{n1} \rho \int_{S_p} \frac{\omega}{j(\omega^2 - \omega_0^2(n1))} \{ F_{11}^{(n1)} I_1 + \sum_p F_{1p}^{(n1)} I_p \} \hspace{1cm} q = 2, 3,$$  \hspace{1cm} (54)

with

$$F_{11}^{(n1)} = \frac{2(\phi_0(n1))_1^2}{(\Psi_0^*(n) \Psi_0(n1))}$$

$$F_{1p}^{(n1)} = F_{p1}^{(n1)} = \frac{2(\tau_0(n1))_p (\phi_0(n1))_1 A}{(\Psi_0^*(n) \Psi_0(n1))}$$.
For the static problem, the terminal relations are defined as

\[
F_{p'p}^{(n)} = \frac{2(\tau_{0(n)}(\xi))_{p}A^{2}}{(\Psi_{0(n)}^{*}, \Psi_{0(n)})}
\]
\[
F_{q'q}^{(n)} = F_{q'p}^{(n)} = \frac{2(\tau_{0(n)}(\xi))_{p}(\tau_{0(n)}(\xi))_{q}A^{2}}{(\Psi_{0(n)}^{*}, \Psi_{0(n)})}.
\]

For the transducer in Fig. 6 the open-circuit normal mode fields are:

\[
V_{q'} = \sum_{p'} z_{q'p'} I_{p'}
\]
\[
p' = 1, 2, 3,
\]

where

\[
V_{1} = \phi_{1}
\]
\[
V_{2,3} = \tau_{2,3} A
\]
\[
I_{2,3} = n_{2,3}
\]

and \(z_{q'p'}\) has the same form as (52).

For the transducer in Fig. 6 with one electrical and two acoustic ports,

\[
\omega_{0(n)} = \frac{n}{l} \pi \left( \frac{C_{44} D_{4}}{\rho} \right)^{1/2}
\]

with

\[
C_{44} D_{4} = C_{44} F + \frac{e_{42}^{2}}{\varepsilon_{s}} \sin^{2} 2\theta.
\]

Evaluation of the coefficients \(F_{q'p}^{(n)}\) gives the impedances

\[
z_{11} = \frac{1}{j\omega C_{0}},
\]

where \(C_{0}\) is the acoustically clamped capacitance

\[
z_{12} = \frac{e_{44}}{j\omega \varepsilon_{s}} \sin^{2} 2\theta
\]

\[
z_{22} = \frac{Z_{0}^{D}}{j} \left( \frac{1}{\xi} + \sum_{n=1}^{\infty} \frac{2\xi}{\xi^{2} - (n\pi)^{2}} \right)
\]

\[
z_{32} = \frac{Z_{0}^{D}}{j} \left( \frac{1}{\xi} + \sum_{n=1}^{\infty} \frac{(-1)^{n}2\xi}{\xi^{2} - (n\pi)^{2}} \right)
\]

where

\[
Z_{0}^{D} = \left( \frac{C_{44} D_{4}}{\rho_{c}} \right)^{1/2}.
\]

The first of these is recognized immediately as equivalent to the results of Berlincourt et al. [16]. In (56b), equivalence is established by noting that [16]

\[
\frac{\varepsilon_{44}}{\varepsilon_{s}} = \frac{g_{44}}{s_{D}}.
\]

By using the expansion formulas [17],

\[
\cot \zeta = \frac{1}{\zeta} + \sum_{n=1}^{\infty} \frac{2\zeta}{\zeta^{2} - (n\pi)^{2}}
\]
\[
\cos \zeta = \frac{1}{\zeta} + \sum_{n=1}^{\infty} \frac{(-1)^{n}2\zeta}{\zeta^{2} - (n\pi)^{2}}
\]

(56c) and (56d) are transformed into the forms

\[
z_{22} = z_{33} = \frac{Z_{0}^{D}}{j \tan \zeta}
\]
\[
z_{23} = z_{32} = \frac{Z_{0}^{D}}{j \sin \zeta}
\]

derived by Berlincourt et al.

It is, of course, clear that the normal mode method is not an efficient method for solving these single wave transducer problems. The advantage of the general analysis is that it allows factors such as the finite area-to-thickness ratio or partial electroding of a transducer to be considered in the analysis. This feature has been exploited very effectively in the design of monolithic, multistage acoustic filters [18].
VIII. NORMAL MODE THEORY OF PIEZOELECTRIC WAVEGUIDES

Starting again from the reciprocity relation (39), an orthogonality relation and mode expansion methods can be developed for waves on piezoelectric waveguides [19]. For the structure shown in Fig. 7, which has a lossless electrical boundary $C_e$ and a lossless acoustic boundary $C_a$, the orthogonality relation is

$$
(\psi_n^*, \psi_m) = 0,
$$

$$
\beta_n^* \neq \beta_m
$$

with

$$
(\psi_n^*, \psi_m) = \int_{S_a} (-v_n^* \cdot T_m - v_m \cdot T_n^*) \cdot dS + \int_{S} (\phi_n^* \{j\omega D_m\} + \phi_m \{j\omega D_n\}^*) \cdot dS.
$$

The forced waveguide is approached in an analogous way to the forced resonator problem, and equations for the mode expansion coefficients $A_n(z)$ are found to have the general form

$$
\left( \frac{\partial}{\partial z} + j\beta_n^* \right) A_n = \int_{C_a} (\psi_n^* \cdot T + v \cdot T_n^*) \cdot N \cdot dl + \int_{S} (\phi_n^* \{j\omega D\} + \phi \{j\omega D_n\}^*) \cdot N \cdot dl
$$

when there are no volume sources present. It should be noted that the forced solution is again determined by applied fields on the boundary, just as in the resonator problem.

This normal mode waveguide theory has been successfully used to analyze the surface acoustic wave amplifier [20] and should have useful application to the interdigital surface wave transducer (Fig. 8). The transducer excitation is provided by the electric boundary integral in (58). In some cases it is desirable to express the excitation as an integration over the waveguide cross section rather than around its boundary. From the divergence theorem, the excitation integral is transformed to

$$
I = \int_{C} (\phi_n^* \{j\omega D\} + \phi \{j\omega D_n\}^*) \cdot N \cdot dl = \int_{S} \nabla \cdot (\phi_n^* \{j\omega D\} + \phi \{j\omega D_n\}^*) \cdot dS.
$$

Use of the identity (38) and the condition

$$
\nabla \cdot D = 0
$$

reduces this further to

$$
I = j\omega \int_{S} (\nabla \phi_n^* \cdot D - \nabla \phi \cdot D_n^*) dS.
$$

If the piezoelectrically induced electric fields of the normal modes are neglected in the first approximation,

$$
\phi = \phi_{\text{applied}}
$$

and

$$
D_n = \epsilon : S_n.
$$

From this, an approximate evaluation of the excitation strength is

$$
I \sim \int_{S} E_{\text{applied}} \cdot \epsilon : S_n^* dS.
$$

Using this relation, estimates can be made of excitation strength for various applied field distributions and crystal orientations. For efficient coupling, the mutual stored energy of the applied electric field and the normal mode strain should be optimized. A more accurate evaluation in strong piezoelectric materials requires consideration of the normal mode electric fields.

IX. PERTURBATION AND VARIATIONAL METHODS

Because the materials used in microwave acoustic applications are usually anisotropic, boundary value problems are much more difficult than those generally encountered in electromagnetic theory and approximation techniques are very important. Perturbation and variational methods have been highly developed in electromagnetism [10] for problems with complicated geometries or anisotropic media, such as ferrites and plasmas. Variational methods have also been effectively applied to acoustic wave problems [7], [23]. In acoustics, solutions for simple geometries are relatively easy to obtain when the media are isotropic, and one of the most
important applications of approximation techniques is in correcting these solutions to take anisotropy into account.

As an illustration, a general perturbation theorem will be developed and applied to the problem of a piezoelectric resonator. The derivation is very similar to that for the reciprocity theorem (Section VI), but the electrical and acoustic sources are now taken to be zero. In (36) perturbed fields (E', etc.) and material parameters (µ', etc.) are used and a time variation e^jw't is assumed. The matrix-type scalar product is then taken with the complex conjugate of the unperturbed fields, giving

\[
v^* \cdot (\nabla \cdot T^*) + T^* : \nabla v' - H^* \cdot \nabla \times E' + E^* \cdot \nabla \times H' = j w' \left( v^* T^* H^* E^* \right) \cdot \left( \begin{array}{c} \rho' \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \psi' \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) - j w \left( v^* \cdot H^* \cdot \nabla \phi' \right) \cdot \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right).
\]

(61)

A second equation is obtained by taking the complex conjugate and interchanging primed with unprimed quantities. These equations are added and, as in Section VI, the sum is reduced to

\[
\nabla \cdot ( - v^* \cdot T^* - v' \cdot T^* + \phi^* \{ j w D' \} + \psi^* \{ j w D \} ) = [ v^* \cdot T^* - \nabla \phi^* ] \cdot \left( \begin{array}{c} \rho' \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \psi' \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) - j w \left( v^* \cdot T^* - \nabla \phi' \right) \cdot \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right).
\]

(62)

where the quasi-electrostatic approximation has been used.

This theorem can be used for the perturbation analysis of either resonator or waveguide problems. Consider a lossless resonator with no electrodes. If (62) is integrated over the resonator volume V, the surface integral vanishes and

\[
\delta \omega = \omega' - \omega = \int_V \left[ \nabla^2 \cdot \phi^* + \frac{\rho}{\epsilon_0} \cdot \phi' \right] dV
\]

(63)

where

\[
\delta \omega = \omega' - \omega
\]

\[
\delta \rho = \rho' - \rho_0, \text{ etc.}
\]

This is an exact expression. Since only the unperturbed field is known, some approximation must be used to evaluate the perturbed fields in the integrals. If the changes in material parameters are very small, the perturbed fields can be replaced by the unperturbed fields, and the perturbed constitutive parameters on the left-hand side replaced by unperturbed values. Otherwise, (63) must be expanded in a series of perturbation orders [24], or the quasi-static modification [10] must be used.

Variational expressions for resonant frequencies and propagation constants are also easily obtained using symbolic notation. Because of the strong analog between the electromagnetic and acoustic field equations, the variational derivations for electromagnetism [10] are easily adapted to acoustic problems.

REFERENCES


