QUARTZ CRYSTAL RESONATOR MODEL PARAMETER SENSITIVITY,  
A FIRST ORDER ANALYSIS  
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ABSTRACT  
The purpose of this paper is to present results of a sensitivity analysis based on quartz crystal resonator model measurements and parameter extraction. Because all measurement systems have random measurement errors, a sensitivity study was conducted to determine the possible bounds of those errors on single mode quartz resonator device model parameters derived from measured scattering (S-) parameters. A first order sensitivity analysis is derived which predicts the variations in the motional arm resistance versus the resonance S-parameter as a function of measurement error. Device simulations were conducted to show the correlation between the first order analysis and the expected results. Device measurements were made using scattering parameters and the model parameters were extracted following the recommended EIA-512 standard. The first order sensitivity analysis, device measurement simulations and experimental results are presented and a discussion is provided.  

Introduction  
An ideal single mode quartz crystal resonator model operated as a one port and a two port device is shown in Fig.1 and Fig.2, respectively. The details of the model development and analysis have been previously discussed by Hafner [1]. These basic models are believed to be valid over the broad operation frequencies of the quartz resonators, however, mounting and packaging techniques can influence the device's terminal impedances. This paper will not discuss these second order effects and the data and analysis presented is believed valid over the actual device operating frequencies.

There are several methods for measurement of quartz crystal resonators. A review of these techniques has been previously presented and will not be discussed [2]. The measurement technique implemented requires the data acquisition of either the device's one port or two port S-parameters. Calibration measurements are made versus frequency using known terminations in a precision device test fixture. These calibration runs are then used to obtain error coefficients which compensate for system errors in the hardware configuration [3]. These errors vary from system to system and from day to day but are assumed stationary over periods of several hours. The device under test (DUT) is mounted in the test fixture and the S-parameters versus frequency are obtained. The device is typically environmentally controlled for exacting measurements. Using the calculated error coefficients, the compensated device S-parameters are obtained. Given the device's terminal property S-parameters, the quartz crystal resonator model parameters can be extracted [4],[5]. There have been numerous reports on the accuracy of actual device model parameter extraction and the results indicate very good reproducibility and accuracy [6],[7],[8]. Model parameters are typically
measured to within a few tenths of a percent or less and series resonance is measured to better than 1 part per million. The question remains as to the quantitative uncertainty to which the model parameters can be stated.

Despite all attempts for exact measurement of the calibration and DUT S-parameters, errors are introduced into the data due to non-stationary, random sources. These errors are caused by various internal and external noise sources, test frequency deviations, temperature variations and others. Because of these errors, exactly repeatable measurements are not possible. In fact, this is the principal motivation for performing a statistical analysis for parameter extraction [4] where the errors are assumed to be random and non-systematic. Examples of actual device parameter extraction from several experimental device data acquisition runs are shown in Table 1 and 2.

The parameters provided were extracted without the use of any data averaging of either the calibration or the DUT S-parameter measurement data and all the consecutive data runs are shown. The measurement techniques are representative of typical results, although actual numbers may vary slightly depending on averaging techniques and the measurement system. A further comment is that $C_0$ and $C_2$ were obtained using the parameter extraction equations [4], which is a non-standard measurement technique. The normal procedure is to extract these capacitance values off-resonance to greater accuracy. This is irrelevant to the study in concern, since the focus is on the resonator's motional arm terms. From the tables it can be seen that the motional arm terms, $R_1$, $L_1$, and $C_1$, have a statistical variation. It is assumed these variations are a result of random errors in the calibration and device measured S-parameters. The following sections will present a first order sensitivity analysis based on the measurement and extraction techniques.

### Table 1. Extracted model parameters from a typical single mode quartz crystal resonator, Device #263.

<table>
<thead>
<tr>
<th>$R_1$(Ohm)</th>
<th>$L_1$(mH)</th>
<th>$C_1$(pf)</th>
<th>$C_2$(pf)</th>
<th>$C_0$(pf)</th>
<th>$f_0$(MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>37.152</td>
<td>206.116</td>
<td>9.596</td>
<td>2.293</td>
<td>.772</td>
<td>.561</td>
</tr>
<tr>
<td>37.114</td>
<td>206.152</td>
<td>9.594</td>
<td>2.141</td>
<td>.372</td>
<td>.592</td>
</tr>
<tr>
<td>37.1</td>
<td>206.159</td>
<td>9.594</td>
<td>2.101</td>
<td>.351</td>
<td>.595</td>
</tr>
<tr>
<td>37.127</td>
<td>206.074</td>
<td>9.598</td>
<td>2.015</td>
<td>.251</td>
<td>.676</td>
</tr>
<tr>
<td>37.108</td>
<td>206.078</td>
<td>9.597</td>
<td>2.019</td>
<td>.236</td>
<td>.694</td>
</tr>
<tr>
<td>37.104</td>
<td>206.079</td>
<td>9.597</td>
<td>1.925</td>
<td>.168</td>
<td>.691</td>
</tr>
<tr>
<td>37.128</td>
<td>206.106</td>
<td>9.597</td>
<td>1.842</td>
<td>.176</td>
<td>.819</td>
</tr>
<tr>
<td>37.098</td>
<td>206.109</td>
<td>9.596</td>
<td>2.004</td>
<td>.243</td>
<td>.675</td>
</tr>
<tr>
<td>37.103</td>
<td>206.106</td>
<td>9.596</td>
<td>1.808</td>
<td>.086</td>
<td>.721</td>
</tr>
<tr>
<td>37.127</td>
<td>205.914</td>
<td>9.605</td>
<td>1.662</td>
<td>.086</td>
<td>.721</td>
</tr>
</tbody>
</table>

**Average** 37.1161 206.0895 9.5969 1.9804 .2741 .6878

**Maximum** 37.152 206.159 9.605 2.293 .772 .592

**Minimum** 37.104 206.079 9.597 1.925 .168 .691

**Std. Dev.** .016257 .06453379 .00296788 .17152154 .18920859 .08930378

**X Std. Dev.** .0438004 .03131351 .03071701 8.6609545 .69.029036 13.016971

### Table 2. Extracted model parameters from a typical single mode quartz crystal resonator, Device #X1.
Parameter Extraction Technique

The parameter extraction technique has been previously presented and the reader is referred to the literature [4]. The sensitivity analysis will first be conducted for the one port model. Development of the two port model will be delayed to the end of the paper and is a simple extension of the one port analysis. Only the one port model concept is discussed in detail. The device input admittance is given by

$$Y = \frac{R - j}{1} .$$

The normalized admittance can also be written in terms of its S-parameters as

$$Y = \frac{1 - S_{11}}{1 + S_{11}} .$$

Equations (1) and (2) couple the measurable S-parameters to the device models Y-parameters. Therefore, by measuring the S parameters as a function of frequency around resonance, the model parameters can be obtained using suitable extraction techniques. This is the basis for the EIA-512 quartz crystal model parameter extraction technique [4]. Given the statistical nature of the parameter extraction technique and the random noise contributions to the S-parameters, the question remains as to the accuracy in which the parameter values can be obtained.

First Order Sensitivity Analysis

One Port Model

In order to determine the quantitative effects of random errors on the extracted parameters, a first order sensitivity analysis was derived [5]. The analysis is conducted narrow-band around the series resonant frequency \( \omega_{02} \) where the reactance, \( X_1 \), is approximately zero \( \{ X_1 \approx 0 \} \) and assumes that the \( |S_{11}| \approx S_{11} \). These are very broad assumptions, however, they greatly simplify the mathematics, the results show the error trends, and the predictions are in good agreement with simulations, as shown in the following sections.

Equation (1) can be written, when \( X_1 \approx 0 \), as

$$Y = \frac{j \omega + 1}{R_1} .$$

where \( \omega = R_1 B_0 \). Then

$$R_1 = \frac{j \omega + 1}{Y} .$$

It is desired to find the sensitivity of \( R_1 \) to changes in \( S_{11} \). This is accomplished by finding the derivative of \( R_1 \) with respect to \( S_{11} \), given by

$$\frac{8R_1}{8S_{11}} = \frac{8R_1}{8Y} \cdot \frac{8Y}{8S_{11}} .$$

Taking the required derivatives yields

$$\frac{8R_1}{8Y} = -\omega + 1$$

and

$$\frac{8Y}{8S_{11}} = \frac{-2}{(1 + S_{11})^2} .$$

Substituting (6) and (7) into (5) yields

$$\frac{8R_1}{R_1} = \omega \frac{8S_{11}}{Y^2 (1 + S_{11})^2} .$$

Now divide by \( R_1 \) and multiply by \( 8S_{11} \) on either side of (9), yielding

$$\frac{8R_1}{R_1} = \frac{2(\omega + 1) S_{11}}{Y^2 (1 + S_{11})^2} .$$

Substituting (4) for \( R_1 \) and using (2) for \( Y \) changes the form of (9) to

$$\frac{8R_1}{R_1} = \frac{2(\omega + 1) S_{11}}{Y^2 (1 + S_{11})^2} .$$

The form of (10) yields the fractional change of \( R_1 \) as a function of \( S_{11} \) and the deviation of \( S_{11} \) from an ideal value. The form of the equation predicts greater fractional changes in \( R_1 \) as the resonator resistance is farther from the characteristic impedance of the measurement system and the change increases rapidly as \( S_{11} \) approaches unity. Also, the fractional change in the nominal value will increase linearly with the random noise introduced in the system.

Since this is only a first order model, the crystal impedance can be further approximated as \( R_1 \) at resonance, \( \{ R_1 \approx R_0 \} \), which can be used in the substitution of \( S_{11} \) in (9), yielding

$$\frac{8R_1}{R_1} = \frac{2}{Y^2 (1 + S_{11})^2} .$$

$$\frac{8R_1}{R_1} = \frac{R_0 (1 + R_1 / R_0)^2}{8S_{11}} .$$

where \( R_0 \) is the characteristic impedance of the measurement system. This equation is only approximate and it will probably give only an order of magnitude estimation of the true deviation. However, it clearly shows the expected trend. As the normalized motional arm resistance increases for a given \( S_{11} \), the maximum deviation in \( 8R_1 \) increases in a nonlinear fashion and is quadratic for large values. As the normalized motional arm resistance approaches zero, the maximum deviation in \( R_1 \) approaches \( R_0 \).
Since the parameter extraction technique is accomplished using many data points to aid in the elimination of the noise, the analysis needs to be extended assuming statistical variations in the parameters. The $1^{st}$ values of $R_1$ and $S_{11}$ are expressed as

$$R_{11} = \overline{R_1} + \delta R_1 \quad \text{or} \quad S_{11} = \overline{S_{11}} - \overline{S_1}$$

and

$$S_{11} = \overline{S_{11}} + \delta S_{11} \quad \text{or} \quad S_{11} = S_{11} - \overline{S_{11}}$$

(12) and (13)

where $\overline{R_1}$ and $\overline{S_{11}}$ are the average values of the resistance and scattering parameters, respectively. Substitution of (12) and (13) into (10) yields

$$\frac{R_{11} - \overline{R_1}}{R_{11}} = \frac{2}{1 - S_{11}^2} (S_{11} - \overline{S_{11}})$$

(14)

Squaring both sides of (14) and taking the variance yields

$$\frac{\sigma_{R_1}^2}{\overline{R_1^2}} = \frac{2}{1 - S_{11}^2} \sigma_{S_{11}}^2$$

(15)

where $\sigma^2$ is the variance. Next taking the square root of (15) yields the form

$$\frac{\sigma_{R_1}}{\overline{R_1}} = \frac{2}{1 - S_{11}^2} \sigma_{S_{11}}$$

(16)

where $\sigma$ is the standard deviation. The form of (16) represents the fractional standard deviation of $R_1$ versus the measured value of $S_{11}$ and the standard deviation of $S_{11}$ due to random noise. As shown in (16), as $S_{11}$ approaches unity, the uncertainty of the measurement of $R_1$ increases very rapidly. Again, the uncertainty in $R_1$ is also directly proportional to the uncertainty in the measurement in $S_{11}$. Fig. 3 is a plot of the first order model predictions for the fractional deviation of $R_1$ versus $S_{11}$ with $\sigma_{S_{11}}$ as a parameter.

Based on these results, some comments on measurements of the resonator devices can be made. First, it is important to characterize the measurement system. It would be advantageous to reduce the system noise as much as possible in the hardware. This may also include remote noise sources which penetrate the system. Secondly, the system noise can be reduced by taking multiple data runs and then implementing suitable signal processing. This will increase the measurement accuracy at the expense of data acquisition time. It would be desirable to have the measurement systems characteristic impedance ($R_0$) closely match the crystal’s resistance. Since this can not typically be accomplished over all crystal’s, some loss of sensitivity will occur. This also indicates that a general upper bound can not be placed on the measurement accuracies of all crystals. The parameter uncertainties will vary with measurement system and with the inherent crystal parameters. Also, the noise can be affected by the system IF bandwidth, the frequency source settling time and the data averaging and processing techniques. Finally, the standard deviation of $S_{11}$ ($\sigma_{S_{11}}$) is most likely a function of $S_{11}$. Therefore, it would be required to quantify this relationship in order to know the measurement uncertainties in measuring devices.

**Resonator Modal Simulations**

In order to confirm the first order sensitivity analysis presented in the previous section, data simulations were conducted on typical quartz resonator parameters. A computer program was written which would input the ideal quartz crystal parameters used in the models of Figs.1 and 2 and then output a set of ideal $S$-parameters versus frequency; typically equi-spaced around the 3dB or 6dB bandwidth. These $S$-parameters simulate the data measurements from an ideal noiseless measurement system. Another software package was written that would input the ideal $S$-parameter data and then add a random perturbation to the $S$-parameters. This simulates an actual data measurement system having a given random noisy environment. The random perturbation is based on a random number generator which has a zero mean and a standard deviation input by the user. A sample distribution is plotted in Fig. 4 superimposed on a calculated gaussian distribution with a standard deviation of 0.01.

Since noise and the random number generator are statistical in nature, it is required to make many runs to obtain the proper results. First it is required to obtain data of $S_{11}$ versus $R_1$ over the ranges of interest. This can be accomplished in many ways. The approach taken was to start with a typical crystal parameter specification, then change $R_1$, $L_1$ and $C_1$ such that the filter $Q$ remained constant and then find $S_{11}$ at resonant frequency. Ten sets of $S_{11}$ data versus frequency were generated in this manner and Table 3...
Fig. 4. Superimposed plot of an ideal gaussian curve with a $\sigma$ of 0.01 versus the random number generated values.

<table>
<thead>
<tr>
<th>$R_1$ (ohms)</th>
<th>$L_1$ (mH)</th>
<th>$C_1$ (pF)</th>
<th>$C_2$ (pF)</th>
<th>$S_{111}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>50</td>
<td>0.0015</td>
<td>5</td>
<td>0.00125</td>
</tr>
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<td>75</td>
<td>145</td>
<td>0.0032</td>
<td>5</td>
<td>0.0015</td>
</tr>
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<td>100</td>
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<td>450</td>
<td>0.0035</td>
<td>5</td>
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<tr>
<td>150</td>
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<td>0.0041</td>
<td>5</td>
<td>0.00101</td>
</tr>
<tr>
<td>200</td>
<td>1000</td>
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<td>5</td>
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<td>500</td>
<td>2600</td>
<td>0.0013</td>
<td>5</td>
<td>0.008189</td>
</tr>
<tr>
<td>1000</td>
<td>3600</td>
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<td>5</td>
<td>0.0094477</td>
</tr>
<tr>
<td>2000</td>
<td>8250</td>
<td>0.00025</td>
<td>5</td>
<td>0.0094472</td>
</tr>
</tbody>
</table>

Table 3. Ideal data generated to simulate $R_1$ variations versus $S_{11}$ for an arbitrarily chosen fixed $Q$.

summarizes the data parameters. In order to simulate the effects of the measurement noise, each set of $S_{11}$ data for each value of $R_1$ was input into the random number generator program twenty-five times with a different 'seed' for each run and with a user input $\sigma_{S_{11}}$. This produced twenty-five data files simulating twenty-five data acquisition runs on an ideal quartz crystal device having a given standard deviation in $S_{11}$. The quartz resonator parameters were extracted [4] and then a spreadsheet was used to obtain the standard deviation and average value of $R_1$ from the twenty-five runs. This procedure was then imposed on each of the ten data sets to generate one complete set of simulation data of $S_{11}$ versus $R_1$ for a given $\sigma_{S_{11}}$.

Fig. 5, 6 and 7 show superimposed plots of the simulations and the first order analysis of the fractional standard deviation of $R_1$ versus $S_{11}$ for various $\sigma_{S_{11}}$. There is good, but not perfect,
agreement between the simulations and the sensitivity analysis. This is to be expected since the first order analysis has made some very broad approximations. The simulations are exact in their implementation and are the standard by which to compare the theory. However, the results show that this first order model is certainly within the correct order of magnitude.

**Experimental Data**

In order to compare the theory to the data, a series of measurements were made on three different crystals. The three resonators were measured as one port devices using an HP 3577 Automatic Network Analyzer (ANA) using a precision test fixture. The ANA was computer controlled using a PC and the data was taken across an IEEE-488 bus for further processing. The ANA was set at a resolution bandwidth of 10Hz which reduced the system measurement noise at the cost of increased measurement time. The data was taken in the CW mode using the internal ANA source. Calibration data was also taken over the frequency bands of interest for a short, open, and 50Ω load. A three term error correction model was implemented to remove stationary systematic errors in the measurement system [6]. Only single data points were taken at each frequency; no data averaging or noise reduction processing was imposed on the DUT data or the calibration data. It should be stated that multiple data points and data processing should be imposed on the calibration runs to reduce the calibration noise as much as possible. No regulation or monitoring of environmental conditions was conducted and this can lead to statistical errors. It is assumed these are small compared to the noise in the system, but were not verified.

Each crystal was measured twenty-five consecutive times at sixteen equi-spaced frequencies around the crystals 6dB bandwidth. The data was then processed through the error correction model to remove the stationary system measurement errors. The data was then input into the parameter extraction program [4] and the crystal model parameters are obtained.

The twenty-five data runs for each crystal are then processed to calculate the average value of $S_{11}$ and $\sigma_{S_{11}}$ at resonance, and the fractional standard deviation of $R_1$. A comparison of the measured data versus the first order sensitivity predictions is shown in Table 4.

There is a scatter in the data and there were only three devices tested, however, there is good correlation between the predictions and measurement for all three crystals. The predictions are of the correct magnitude; their absolute values are not exact owing to the approximations in the theory. More data over varying different crystals and parameters is necessary to provide further correlation.

| DEVICE NAME | $|S_{11}|^*$ | $\sigma_{S_{11}}$ | $\left(\frac{\sigma_{S_{21}}}{R_1}\right) \times 100$ |
|-------------|-------------|-----------------|-----------------------------------------------|
| J69         | .157        | .00096          | .4218                                         |
| CC4         | .182        | .00044          | .0415                                         |
| XX1         | .947        | .00013          | .4763                                         |

Table 4. Comparison of theory and experimental data of three different quartz crystal resonator devices.

**Extension of the Analysis to the Two Port Resonator Model**

It has been previously shown that the two port crystal resonator parameters of Fig. 2 can be easily obtained from two port measurements [4]. The model parameters are obtained by removal of the parasitic elements on the shunt arms of the two ports, creating a modified value of $S_{21}$. The $S_{21}$ data is then used to obtain the model parameters similar to the one port model. The two port first order sensitivity analysis is similar to the one port with the simple substitution of $S_{21}$ for $S_{11}$, yielding

$$\frac{\sigma_{R_1}}{R_1} = \frac{2}{1-S_{21}^2} \sigma_{S_{21}}$$  \hspace{1cm} (17)

The two port analysis proceeds as previously discussed. In general, the two port analysis is advantageous since it eliminates some parasitic elements and is generally more accurate owing to the calibration and measurement techniques but requires the taking of more data, more data processing and, therefore, more time.

**Discussion and Conclusions**

Based on the results of the previous sections, some general comments can be made:

1) As $|S_{11}|$ approaches unity, the uncertainty in the extracted model parameters will increase rapidly. This suggests that the measurement system characteristic impedance should be close to the crystal resonance for greatest accuracy.

2) As $\sigma_{S_{11}}$ increases, (i.e., as the system measurement noise increases), errors in the measured DUT parameters will also increase linearly. This implies that the uncertainty in the quoted model parameters is related to the system under which the device was measured.

3) Multiple data point acquisition at each frequency and post processing will reduce
and thereby decrease the uncertainty of the measured DUT parameters. Further, the calibration data should be averaged and processed in a manner to give the highest degree of accuracy possible so as not to influence the DUT data acquisition.

4) Frequency set point error or jitter will result in an increase in $\sigma_{S_{11}}$.

5) Random environmental fluctuations will increase the effective noise, however, systematic environmental fluctuations will need to be extracted from the data for accurate extraction of the model parameters.

Further work on the sensitivity analysis is certainly warranted. The analysis is really only valid where $R_t \gg X_1$. This is only valid in close proximity to series resonance and is not even valid over the entire 3 dB bandwidth of the device. This would suggest that a second order model valid over a broader bandwidth needs development. This is currently under investigation. Because of the analysis approach, nothing is derived of the sensitivity of $L_1$ and $C_1$. This is again owing to the extremely narrowband approximation of the analysis. However, it has been previously shown through simulations that $L_1$ and $C_1$ have the same general characteristic sensitivity [4], therefore, the analysis can be inferred to apply within the correct magnitude for $L_1$ and $C_1$.

A second order model is desirable which is valid over the 3dB or 6dB bandwidth of the device. The model should include variations in both the magnitude and the phase of $S_{11}$ which will yield sensitivities of all the resonator model parameters and will be valid over the full range of $S_{11}$. This model will set accurate limits of uncertainties or model parameters for a given measurement system.

This work suggests that measurement and calibration of the measurement system is necessary in order to determine the uncertainties in the extracted resonator model parameters.

This paper has presented the results of a first order sensitivity analysis for quartz crystal resonators. A relationship is derived for the fractional change of $R_t$ versus $|S_{11}|$ and the standard deviation of $S_{11}$ ($\sigma_{S_{11}}$) for the one port model and a similar relation is given for the two port model.

Simulations of ideal crystal resonator measurements with the inclusion of random noise on the magnitude of $S_{11}$ show good correlation between the first order sensitivity analysis and the simulations versus $S_{11}$. A comparison of experimentally measured data and theory also show good correlation for the three crystals presented.

Acknowledgements

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References


