Ray-Optical Evaluation of $V(z)$ in the Reflection Acoustic Microscope

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Abstract—When viewing materials having Rayleigh velocity greater than the velocity in the coupling fluid, the output voltage of the reflection acoustic microscope varies with defocus distance as a result of coupling to the Rayleigh waves at the object surface. In this analysis, the fields reflected by the surface are separated into a geometrically reflected part, and one due to the excitation and reradiation of the Rayleigh wave. Using a ray-optical approach, simple formulas are derived for the contribution to the output voltage from the geometrically reflected and Rayleigh fields. The formulas give good overall agreement with measurements. Because the results are in analytic form, it is easily seen how the various geometric and acoustic parameters of the lens and object affect the output voltage.

I. INTRODUCTION

Working in the reflection mode, the acoustic microscope with water couplant has been used to image the surfaces of solids, for example, in the study of integrated circuits [1]–[4], and in the investigation of grain structure in metals [5], [6]. For acoustic waves in water, the reflection coefficient at the surface of most metals and crystalline materials is greater than 0.8. Because the reflection coefficient is so high, its variation from point to point on the surface is small and is not sufficient to produce the contrast observed in the images of integrated circuits and metals grains. It has been recognized that the contrast in these images is due to the sensitivity of the microscope's output voltage to height variations [5], [7], [9].

In studying the contrast mechanism, it was found that as the microscope is moved towards an object with a smooth surface, the output voltage exhibits a series of minima and maxima, which was dubbed the acoustic material signature (AMS) [7]. The displacement of the microscope between minima was found to be related to the Rayleigh wave velocity of the object [10], [11]. While these observations were originally made on spherical lenses, they have also been observed with cylindrical lenses, whose focus is a line rather than a point [12], [13].

Two equivalent explanations of the AMS were given in terms of Rayleigh critical angle phenomena [14], [15]. From a ray-optical viewpoint, the AMS is due to interference between the fields of near-axial rays reflected from the surface, and the fields along ray paths that include the excitation and reradiation of the leaky Rayleigh wave [14]. Alternatively, Fourier optics has been used to explain the AMS in terms of the rapid variation of the phase of the reflection coefficient $R(\theta)$ for angles of incidence $\theta$ in the vicinity of the Rayleigh critical angle $\theta_R = \sin^{-1}(V_w/V_R)$, where $V_w$ and $V_R$ are the wave velocities for water and for the Rayleigh wave [15]. The connection between the rapid variation of the phase of $R(\theta)$ near $\theta_R$ and the excitation of the Rayleigh wave is discussed in [16], [17].

In this paper, we employ ray optics to develop simple analytic expressions giving the variation $V(z)$ of the output voltage with displacement $z$ of the object surface from the focal plane. The previous ray-optical study of the AMS [14] predicted the spacing $\Delta z$ of the minima of $V(z)$ based on changes in phase along the different ray paths. The study did not compute the absolute phase or the amplitudes, so that it was not possible to predict the shape of the curve $V(z)$, including the depth of the minima. Fourier optics has been used to accurately predict the $V(z)$ dependence [8], [18], [19]. However, because the Fourier transforms are carried out numerically, one cannot easily see from this approach how the various geometric and acoustic parameters influence $V(z)$.

II. SEPARATION OF GEOMETRIC AND LEAKY WAVE EFFECTS

The reflection coefficient for waves incident on a substrate having Rayleigh velocity greater than about twice that of the couplant is similar to that shown in Fig. 1 for yttrium iron garnet (YIG) and water. The magnitude and phase of $R(\theta)$ have some variation in the vicinity of the longitudinal and shear critical angles $\theta_L$ and $\theta_S$. For $\theta > \theta_L$, $R(\theta) = 1$ for materials with no acoustic damping, but the phase of $R(\theta)$ undergoes a rapid change of $2\pi$ as $\theta$ increases past $\theta_R$. 

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wave. If the Rayleigh critical angle.

shown in Fig. 2. Other poles and zeros lie far from the real

in the complex

k, propagating on a free surface, and

exp

w/VR.

is the radian frequency of the harmonic time dependence

for that in the denominator.

and

in that region is

B. Extraction of the Leaky Wave Poles

As discussed above, the behavior of

is the product of

and the factor in (2), i.e.,

In the absence of dissipation,

will have the same magnitude as

R(kx)

will be nearly the same

The phase of

and

has branch point singularities

is the sum

where the transverse wavenumber

k_x = k_w \sin \theta

In (1), \rho_w

and \rho_s

are the mass densities of the couplant (water) and the

substrate, and

k_t = \omega/V_t,

k_x = \omega/V_x

and

k_w = \omega/V_w

where

is the radiator frequency of the harmonic time dependence

exp (-itwt),

V_t

and

V_w

are the longitudinal and shear wave

velocities of the substrate, and

V_s

is the acoustic velocity of the
couplant.

It is seen from (1) that

R(k_x)

has branch point singularities

at \pm k_t, \pm k_s, and \pm k_w.

In addition it has off axis poles and

zeros, two of which \pm k_p and \pm k_s lie close to the real

k_x

axis in the complex

k_x

plane. The location of the singularities is

shown in Fig. 2. Other poles and zeros lie far from the real

k_x

axis and do not significantly influence the field reflected

from the surface.

In the absence of acoustic dissipation in the solid, the square

bracket in (1) vanishes at the Rayleigh wavenumber

k_R = \omega/V_R.

For

k_x

real and

k_x

> k_s,

the term in the square

bracket is real, while the second term in the denominator

is small and imaginary. Thus, the denominator of

R(kx)

will vanish at complex points

k_x = \pm k_p

close to

k_R,

where

k_p = \beta + i \alpha.

It is found that

\alpha/k_R \ll 1

and that

(\beta - k_R)/k_R \ll \alpha/k_R

[16]. Because the second term in the numerator in (1)

is of opposite sign to that in the denominator,

R(kx)

will have a zero at the points

k_x = \pm k_0,

where

k_0

is the conjugate of

k_p.

Because the poles and zeros are so close to

\pm k_R,

the variation of

R(kx)

for

k_x

near

k_R

will be approximately given by

\frac{k_x - k_0}{k_x - k_p} \approx \frac{k_x + k_0}{k_x + k_p} = \frac{k_x^2 - k_0^2}{k_x^2 - k_p^2}.

This expression has unit magnitude and undergoes a phase

shift of nearly -2\pi as

|k_x|

increases past

k_R,

i.e., as |\theta|

increases past

\theta_R.

Thus, the presence of the pole-zero pair accounts for the major features of the reflection coefficient near

the Rayleigh critical angle.

In the absence of dissipation in the solid, \alpha will be nonzero

because of the leakage of energy into the water. Dissipation in

the solid will cause an additional attenuation of the Rayleigh

wave. If \alpha_d is the attenuation constant of a Rayleigh wave

propagating on a free surface, and \alpha_l is the attenuation due to

radiation into the water in the absence of dissipation in the

solid, then to first order \alpha is the sum \alpha_d + \alpha_l. Since \beta \approx k_R,

we have

k_p \approx k_R + i(\alpha_d + \alpha_l).

It has been found that the effect of dissipation in the solid on

the zero

k_0

of

R(kx)

is to increase its imaginary part by nearly

the same amount as the increase of

Im

(\alpha_d + \alpha_l) [16], [20], [21].

Thus, we also have

k_0 \approx k_R + i(\alpha_d - \alpha_l),

which reduces to

k_p^0

when \alpha_d = 0.

For critical damping

\alpha_d = \alpha_l

so that the zero lies on the real axis, giving zero reflected

field for a plane wave incident at the Rayleigh critical angle

[16], [20], [22].

C. Separation of the Reflected Field

Consider a field incident on the substrate, as shown in Fig. 3.

The field is assumed to have no variation along y, and to be

described by a family of rays that have a line focus at

(x, z = 0).

Provided that z_i is larger than about one wavelength

\lambda_w = 2\pi/k_w,

the pressure along the surface of such an incident field is given by

p_i(x, z_i) = G(\theta) \exp (-ik_w \sqrt{x^2 + z_i^2}) \frac{\sin k_w (x^2 + z_i^2)^{1/4}}{k_w^{1/2} (x^2 + z_i^2)^{1/4}},

where

G(\theta)

is a slow varying amplitude function. A two-
dimensional field is treated here for simplicity, since the ray-

optical expressions derived for the excitation of the leaky
wave can then be applied to the microscope case using the procedures of ray optics.

Let \( P_i(k_x, z_i) \) be the Fourier transform of the incident pressure field \( p_i(x, z_i) \) given in (7). The reflected pressure field \( p_r(x, z_i) \) along the surface can then be found by taking the inverse transform of the product of \( P_i(k_x, z_i) \) and \( R(k_x) \). Using the approximation (6) for \( R(k_x) \), we can separate the pressure of the reflected field into a geometrical part \( p_G(x, z_i) \) and a leaky wave part \( p_L(x, z_i) \).

Thus,

\[
p_r(x, z_i) = p_G(x, z_i) + p_L(x, z_i),
\]

where

\[
p_G(x, z_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_0(k_x) P_i(k_x, z_i) e^{ikx} d k_x,
\]

and

\[
p_L(x, z_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k_0^2 - k_x^2}{k_x^2 - k_p^2} R_0(k_x) P_i(k_x, z_i) e^{ikx} d k_x.
\]

The branch point singularities of \( R_0(k_x) \) in (9) give rise to lateral ray contributions to \( p_G(x, z_i) \) [23], [24]. Ignoring these contributions, the steepest descent evaluation of (9) yields the ordinary ray-optical representation of \( p_G(x, z_i) \), in which the reflected ray field is found as the product of the incident ray field and the local reflection coefficient [23], [24]. Thus, the total reflected field contains a geometrical optics term

\[
p_G(x, z_i) = R_0(k_w \cos \theta) p_i(x, z_i).
\]

The field incident in the vicinity of the point \( x_i < 0 \), at which the ray incident at the angle \( \theta_R \) intersects the surface, will excite the leaky Rayleigh wave propagating to the right.

In the Appendix, it is shown that to the right of the excitation region about \( x_i \), the field \( p_r(x, z_i) \) along the interface associated with the leaky wave propagating to the right is given by

\[
P_r^L(x, z_i) = -2\alpha_i \sqrt{\frac{2\pi |z_i|}{k_w (\cos \theta_R)^3}} e^{-im/2} e^{ik_p(x-x_i)} p_i(x_i, z_i).
\]

The leaky wave fields at a finite distance above the surface in Fig. 3 can be found by tracking along the parallel family of
\(-f < z < -f/2\), the rays reflected from the object surface are refracted by the lens before crossing the \(z\) axis. In this case, the refracted rays cross the \(z\) axis at a focus located a distance \(u\) above the lens surface. For all values of \(z\), the value of \(u\) is found from the imaging formula for a spherical surface to be
\[
u = f(f + 2z)/(2n z),
\] (13)
which gives the virtual focus \((u < 0)\) for \(-f/2 < z < 0\), and a real focus \((u > 0)\) for \(-f < z < -f/2\).

The rays returning to the transducer plane illuminate a circular spot of radius \(r_T\), which is defined by the limiting ray refracted at the edge of the lens, as shown in Fig. 4. For \(|z|\) small and \(R_T > u\), the illumination spot will be smaller than the transducer. In this case, integration of the field over the transducer to obtain the output voltage will be limited to the radius \(r_T\) of the illuminating spot. For \(r_T > R_T\), the integration must be taken over the entire transducer. To find \(r_T\) for \(|z|\) small, we approximate \(\hat{u}\) in Fig. 4 by \(a(f + 2z)/f\), recalling that \(z < 0\). Then for \(u < 0\),
\[
r_T = \hat{u}(D - u)/(-u) = a[1 - 2z(nD - f)/f^2].
\] (14)

The condition \(r_T = R_T\) is obtained at the object displacement
\[
\text{min} = -f^2 \left(\frac{(R_Ta - 1)}{2n(D - f/u)}\right).
\] (15)

For \(z < \text{min}\), the entire transducer is illuminated by the returning field, while for \(z > \text{min}\) only a limited area of radius \(r_T\) is illuminated. As suggested in Fig. 4, when \(|z|\) is not near zero, the reflected rays illuminating the transducer are produced by incident rays that originate from a small area at the center of the transducer.

**B. Ray Fields for \(z < 0\)**

The transducer is assumed to excite fields that reach the lens surface with unit amplitude stress \(T_{zz}\). The wave is transmitted through the lens surface with transmission coefficient \(T_1\). For an uncoated sapphire lens, and water couplant, the variation of \(T_1\) across a lens of 50° half angle is about 2 dB [26]. If a quarter wave matching layer is deposited on the lens surface, the variation of \(T_1\) is still approximately 2 dB for a 50° half angle lens, but the variation increases rapidly for half angles above 50° [26]. In this analysis, we treat \(T_1\) as a constant. The error introduced by this approximation decreases rapidly as the microscope is moved towards the object, since the reflected rays illuminating the transducer cross the interface in a small region near the center of the lens. For the same reason, the reflection coefficient at the object surface is taken to be the value at normal incidence \(R_0(0)\) for all rays, and the transmission coefficient \(T_2\) from water back into the lens is taken to be a constant.

The amplitude of the pressure field in the water is affected by the convergence, and subsequent divergence of the rays. To conserve power, the field amplitude along a ray bundle must vary inversely as the square root of the cross-sectional area of the bundle [25]. Thus, the field returning to the lens surface differs from the field transmitted into the water by the factor \(f/[f + 2z]\). The field returning to the transducer from the lens surface is further reduced by the factor \(|u|/(D - u)\) due to ray spreading in the lens rod. An additional reduction in the field amplitude due to attenuation in the water is given by the factor \(\exp(-2\alpha_u(f + z))\), where \(\alpha_u\) is the attenuation constant in water.

Due to the spherical curvature of the phase fronts of the field illuminating the transducer, the phase of the field arriving at the transducer is equal to the sum of the phase \([2k_w(D/n + f + z)]\) of the central ray, and a curvature correction, which for \(D >> R_T\) is given by \(k_w^2/[2n(D - u)]\). Note that \(k_w/n\) is the wavenumber of longitudinal waves in the lens rod. When the rays reflected from the object surface pass through the focus along the axis, the fields experience an additional phase change of \(-\pi\) [25].

When taken together, the various amplitude and phase factors described above give the following expression for the field at the transducer:
\[
AR_0(0) \exp\left[i(k_w - \alpha_u)2z\right] \frac{f}{|f + z|} \cdot \frac{|u|}{|D - u|} \exp\left[ik_w^2z^2/2n(D - u)\right],
\] (16)
where
\[
A = -T_1T_2 \exp\left[i2k_w(D/n + f) - 2\alpha_wf\right].
\] (17)

**C. Ray Fields for \(z > 0\)**

The ray structure for the case \(z > 0\) is shown in Fig. 5. The focal point for the rays returning to the transducer now lies above the lens surface a distance \(u\) given by (13). For \(z\) small, \(u\) will be greater than \(D\) so that the focal point lies above the transducer, and the ray field is convergent at the transducer, rather than divergent as shown in Fig. 5. The condition \(u = D\) with...
is obtained for \( z = z_0 \), where from (13)

\[
 z_0 = \frac{f^2}{2n(D - f/n)}.
\]

(18)

For this condition, the focal spot lies on the transducer, and the integration of the ray-optical field over the transducer will vanish. Accounting for diffraction, the focal spot will have finite size so that the integration, and hence the output voltage, will have a nonzero minimum.

As \( z \) increases above \( z_0 \), \( u \) decreases below \( D \) and the rays reaching the transducer will be divergent, as shown in Fig. 5. Further increase in \( z \) causes an increase in the radius \( r_f \) of the illuminated spot on the transducer plane, until at some value \( z = z_{\text{max}} \) the entire transducer is illuminated. For \( D \gg a \), it is seen from Fig. 5 that \( r_f \) can be approximated as

\[
r_f = a|D - u|/u.
\]

(19)

Equating \( r_f \) and the transducer radius \( R_T \), and using (13), gives \( z_{\text{max}} \) as

\[
z_{\text{max}} = \frac{f^2(1 + R_T/a)}{2nD - (1 + R_T/a)f/n}.
\]

(20)

For \( z > z_{\text{max}} \) the transducer is fully illuminated.

Using the same arguments as given in the previous section, the reflected ray field at the transducer is again found to be given by (16), except for an additional \( -\pi \) phase shift for \( z > z_0 \) that the fields acquire as they pass through the second focus at \( u \).

D. Output Voltage

The output voltage \( V_G \) produced by the geometrically reflected ray fields falling on the transducer is found by integrating (16) over the transducer. For \( z < z_{\text{min}} \) and for \( z > z_{\text{max}} \) the transducer is fully illuminated and the upper limit of the integration over \( \rho \) must be taken as \( R_T \). Otherwise, the illumination is limited to a circle radius \( r_f \) given by (14) or (19).

We assume that the output voltage that would be produced by a uniform field of unit amplitude is \( V_0 \). Carrying out the integration over the transducer, as discussed above and simplifying, leads to the expression

\[
 V_G = V_0 A R_0(0) \left[ -\frac{2z_0F}{\pi z} \right] (\sin X) \cdot \exp \left[ (ik_{\omega} - \alpha_{\omega})2z + iX \right].
\]

X = \frac{\pi}{2F} \left\{ \begin{array}{ll}
 z(z - z_0), & z < z_{\text{min}}, \text{ or } z > z_{\text{max}},
\end{array} \right.

\begin{align*}
 &\left( \frac{a}{R_T} \right)^2 \frac{z(z - z_0)z^2}{z_0^2(1 + 2z/f)^2}, & 0 < z < z_{\text{max}},
\end{align*}

(22)

\[
 F = n(D - f/n)\lambda_w/R_T^2.
\]

(23)

The quantity \( f/n \) is the focal length inside the lens rod. Thus \( D - f/n \) in (23) represents the distance from the transducer to the back focal plane of the lens. Since \( n\lambda_w \) is the wavelength in the lens rod, \( R_T^2/(n\lambda_w) \) is the Fresnel distance for the transducer. Thus \( F \) in (23) represents the ratio of the separation between the transducer and back focal plane to the Fresnel distance of the transducer. In typical lens designs, \( F \) is near unity.

Making use of (15), (18), (20), and (23), and assuming \( D \gg f/n \), as is frequently the case, we can express \( z_0 \), \( z_{\text{min}} \), and \( z_{\text{max}} \) as

\[
z_0 = (f/R_T)^2 \lambda_w/(2F),
\]

\[
z_{\text{min}} \approx -z_0 \left( (R_T/a) - 1 \right),
\]

\[
z_{\text{max}} \approx z_0 \left( (R_T/a) + 1 \right).
\]

(24)

The value of \( R_T \) is typically chosen to lie between \( a \) and \( 2a \), while \( f \) is somewhat larger than \( a \). Thus, for \( F \) near unity, it is seen from (24) that \( z_0 \), \( z_{\text{min}} \), and \( z_{\text{max}} \) are all on the order of \( \lambda_w \), which is usually much smaller than \( f \). As a result, \( 2z/f \ll 1 \) for \( 0 < z < z_{\text{max}} \), and the middle expressions for \( X \) in (22) holds approximately over the entire range \( z_{\text{min}} < z < z_{\text{max}} \).

The variation of \( |V_G(z)| \) is shown in Fig. 6 for \( F = 1 \) and two values of \( R_T/a \). Note that for given values of \( a \) and \( F \), doubling \( R_T \) implies that \( D \) must be increased by a factor of four. For \( |z| \gg z_0 \), \( \sin X \approx 1 \) so that the variation in \( |V_G(z)| \) is due to the factor \( (1/Z) \) in (21), and the attenuation in water \( \exp(-2a_{\omega}z) \), which has been neglected in drawing Fig. 6. The zero of \( V_G(z) \) at \( z = z_0 \) results from the focusing of the reflected rays onto the transducer by the lens. In measurements made with various spherical and cylindrical lenses, a minimum of \( |V_G(z)| \) occurs just to the right of the peak, as suggested in Fig. 6 [5], [8], [12], [27].

The choice \( F = 1 \) places the lens at the point where the radiated fields change from near field to far field dependence. As a result, diffraction effects will be significant in the propagation between the lens and transducer. Diffraction, together with variation in \( T_1 \), \( T_2 \), and \( R_0(\theta) \) that have been neglected, are expected to have the effect of reducing the voltage for \( z \) small. However, spherical divergence of the reflected rays for \( |z| \gg z_0 \) reduces the significance of these effects. The zero at \( z = 2z_0 \) for \( R_T = a \) results from phase cancellation across the transducer. Unlike the one at \( z = z_0 \), the existence of the zero at \( z = 2z_0 \), and even its location, will be sensitive to the exact size of the transducer, variations of \( T_1 \), \( T_2 \), and \( R_0(\theta) \), and to diffraction effects.

When \( F < \frac{1}{2} \), the term \( \sin X \) in (21) has one or more zeros in the range \( z < 0 \). These zeros are the result of phase cancellation across the transducer. This effect is shown in measurements made by I. Smith [28] with a 12.5 MHz fused quartz lens whose aperture had been reduced to eliminate Rayleigh wave excitation on Dural. The measured response for a Dural object, shown in Fig. 7, is therefore due entirely to the geometrically reflected field. The deep minimum for \( z < 0 \) is con-
Fig. 6. Output voltage $V_G(z)$ due to the geometrically reflected fields for a lens with $F = 1$. The solid curve is for $R_T = \alpha$. If $R_T = 2\alpha$, $V_G(z)$ is modified in the range $-1 < z < 3$ as shown by the dashed curve, but is unchanged outside this range.

Fig. 7. Output voltage for a 12.5 MHz, limited aperture lens with $F = 0.36$ showing a zero of $\sin X$ for $z < 0$.

consistent with the short length of the lens rod compared to the Fresnel distance. The dimensions of the lens used to make the measurement in Fig. 7 give the parameters $f = 21.4$ mm, $F = 0.36$, and $z_0 = 0.532$ mm from (18). The stop radius was not recorded when the measurements were made. Reasonable agreement was obtained for $a = R_T/3$ and the resulting curve is shown dashed in Fig. 7.

IV. Voltage Due to the Leaky Wave Field for $z < 0$

The transducer and lens geometry is shown in Fig. 8, together with the rays relevant for computing the field at the transducer resulting from the leaky wave excited on the object surface.

A. Structure of the Rayleigh Rays

As it propagates along the surface, the leaky Rayleigh wave in the plane of the drawing reradiates at the angle $\theta_R$, as shown in Fig. 8. One of these rays, called the principal ray, appears to come from the focus, and is refracted at the lens so
as to travel parallel to the lens axis. Rays that are reradiated on either side of the principal ray in the plane of the drawing travel parallel to it in the water. At the lens surface they are refracted so as to focus inside the lens at distance \( f/n \) \( \cos^2 \theta_R \) from the lens surface. Since the lens has cylindrical symmetry about the \( z \) axis, the focus of the entire leaky wave family is in the form of a ring of radius \( \rho_R \approx f \sin \theta_R \).

As seen in Fig. 8, those rays radiated to the right of the principal ray pass through the ring focus and illuminate the portion of the transducer to the left of the principal ray. Provided that the lens aperture \( a \) is not close to \( \rho_R \) and, assuming \( D \gg a \), these rays will illuminate the entire portion of the transducer to the left of the principal ray. Considering the entire cylindrically symmetric family, it is seen from Fig. 8 that each point of the transducer is illuminated by two rays, one from each side of the ring focus. One set of rays crosses the axis, which is a line focus for the set.

### B. Ray Fields

As in the case of the geometrically reflected rays, the lens is assumed to be illuminated by a family of rays parallel to the axis and carrying a stress field \( T_{zz} \) of unit amplitude. This field is transmitted into the water with transmission coefficient \( T_1 \). Because of the spherical convergence, the field of the ray incident on the object surface at the angle \( \theta_R \) will have an amplitude increased by the factor \( f/(f/n) \cos^2 \theta_R \). Attenuation in the water results in the additional factor \( \exp [-\alpha_w(f - |z| \sec \theta_R)] \). The phase at the surface of the ray incident at \( \theta_R \) is equal to the phase \( k_w(f + D/n) \) at the focus in the absence of the substrate, less the phase change \( k_w |z| \sec \theta_R \) for propagation from the surface plane to the focus. Using the foregoing, the field at the surface of the ray incident at the angle \( \theta_R \) is

\[
p_i = T_1 \frac{f}{|z| \sec \theta_R} \exp [-\alpha_w(f - |z| \sec \theta_R)] \exp [ik_w(f + D/n - |z| \sec \theta_R)]. \tag{25}
\]

The field given by (25) excites the leaky Rayleigh wave in the vicinity of the radius \( |z| \tan \theta_R \), which then propagates cylindrically inward towards the \( z \) axis. After passing the focus at the \( z \) axis, with the attendant phase change \( \exp (-i\pi/2) \) appropriate to a line focus [25], the leaky wave diverges. When it again reaches the radius \( |z| \tan \theta_R \), it launches the principal ray of Fig. 8, whose initial field is given by (12), with \( x = x_1 \) replaced by \( 2|z| \tan \theta_R \), and multiplied by the factor \( \exp (-i\pi/2) \).

As discussed in the Appendix, expression (12) gives the initial field of the principal ray provided that \( z < z_1 \), where

\[
z_1 = -\lambda_w/(4 \cos \theta_R \sin^2 \theta_R). \tag{26}
\]

For \( |z| < |z_1| \), the initial field of the principal ray is modified by the factor \( [1 - (1/2) \text{erfc} (z)] \). For example, when \( z = 0 \), then \( z = 0 \) for the principal ray and the factor is \( 1/2 \) or 6 dB.

The field propagating along the principal ray back to the lens surface decreases in amplitude by the factor \( (|z| \tan \theta_R / \rho_R)^{1/2} \) due to cylindrical spreading of the rays (out of the plane of the drawing in Fig. 8). Water attenuation introduces the additional amplitude factor \( \exp [-\alpha_w(f - |z| \sec \theta_R)] \). Finally, transmission through the interface is accounted for by the coefficient \( T_2 \).

The rays in the lens rod experience convergence and divergence in the plane of the drawing of Fig. 8, as well as in the radial direction. The convergence to the ring focus, and subsequent divergence, in the plane of the drawing effects the field reaching the transducer by the amplitude factor \( f \cos^2 \theta_R / (nD) \) \( (25) \), for \( D >> f/n \). The focus introduces the phase factor \( \exp (-i\pi/2) \) corresponding to a line focus. The radial spreading effects the field at the transducer by the amplitude factor \( \rho_R / \rho \) \( (25) \), where \( \rho \) is the radial distance at which the ray intercepts the transducer plane. An additional phase factor \( \exp (-i\pi/2) \) must be included for those rays that cross the lens axis.

The phase change due to the path length along the principal ray from the object surface to the transducer is given by

\[
k_w(f + D/n) \sec \theta_R \] for propagation from the focus to the object surface. Because neighboring rays also pass through the ring focus, they will have the same phase except for a correction due to phase front curvature. For rays that do not cross the lens axis the correction is given by \( k_w(\rho - \rho_R)^2/(2nD) \). Rays that cross the axis have phase correction \( k_w(\rho + \rho_R)^2/(2nD) \).

The various factors described in the preceding three paragraphs can be combined into a single factor that gives the variation of the field in going from the object surface to the transducer plane. This factor is given by

\[
T_2 \left( \frac{|z| \tan \theta_R}{\rho_R} \frac{f \cos^2 \theta_R}{nD} \frac{\rho_R^2}{\rho} \right)^{1/2} \exp [-\alpha_w(f - |z| \sec \theta_R)] \exp [-2k_w |z| \sec \theta_R] \exp \left[ ik_w(\rho - \rho_R)^2/(2nD) \right] + \exp [-2k_w(\rho + \rho_R)^2/(2nD)]. \tag{27}
\]

In this expression, the first term in the bracket arises from the rays that do not cross the lens axis, while the second term comes from rays that do cross the axis.

The field at the transducer is found by multiplying (27) by (12), with \( p_i \) substituted from (25), and finally by the factor \( \exp (-i\pi/2) \). After some manipulation the field is found to be

\[
A(2\alpha_k) \frac{(2\pi f^2 \sin \theta_R)^{1/2}}{nk_wD} e^{i3\pi/4} \exp [2(\alpha_w - \alpha \sin \theta_R) |z| \sec \theta_R] \exp [-i2k_w |z| \cos \theta_R] \frac{1}{\sqrt{\rho}} \exp \left[ ik_w(\rho - \rho_R)^2/(2nD) \right] + \exp \left[ ik_w(\rho + \rho_R)^2/(2nD) \right], \tag{28}
\]

where \( A \) is defined in (17).
C. Leaky Wave Voltage $V_L$

The output voltage due to the leaky wave is found by integrating the field (28) over the transducer. Again we let $V_0$ be the voltage that would be produced by a uniform illumination of unit amplitude. Changing the variable of integration, the voltage can be written as

$$V_L = V_0 A K \left[ \frac{2 \alpha_i \lambda_w \left( \sqrt{n f^3 \sin \theta_R} \right)^{1/2}}{(n \lambda_w D)^{3/4}} \right] e^{i \pi} \cdot \exp \left[ -i 2 k_w |z| \cos \theta_R \right] \cdot \exp \left[ 2 (\alpha_w - \alpha \sin \theta_R) |z| \sec \theta_R \right],$$

(29)

where

$$K = e^{-i n/4} \frac{2}{\eta_T} \int_0^{\eta_T} \left[ e^{i(n-\eta R)^2} + e^{-i(n+\eta R)^2} \right] \sqrt{\eta} \ d\eta.$$

In (30)

$$\eta_T = \sqrt{\frac{k_w}{2 n D}} \frac{R_T}{F} \approx \sqrt{\frac{\pi}{F}}$$

and

$$\eta_R = \sqrt{\frac{k_w}{2 n D}} \frac{\rho_R}{F} \approx \sqrt{\frac{\pi}{F}} \left( \frac{\rho_R}{R_T} \right).$$

(31)

The integral in (30) cannot be evaluated in terms of simple functions. We have computed $K$ numerically as a function of $\rho_R$ for various values of $F_T$, i.e., various values of $F$. The results of the calculations are shown in Fig. 9. It is seen that neither the magnitude or phase of $K$ is sensitive to $\rho_R$.

For $z < z_1$, the only dependence of $V_L$ on $z$ is through the phase term $\exp (-i k_w |z| \cos \theta_R)$ and the exponential amplitude term. The dependence of $|V_L|$ on $z$ is shown in Fig. 10 for a fused quartz object ($\theta_R = 26^\circ$) and a lens with $F = 1$ and $a = f \sin 45^\circ$. Curves are shown for two values of transducer radius $R_T = a$ and $R_T = 2a$. The difference in the slopes of the two curves for $|V_L|$ is due to the fact that the abscissa is $z/z_0$, where for a fixed value of $F$ it is seen from (24) that $z_0$ varies with $R_T$. Had the curves been plotted as a function of $z$, the slopes would be the same.
With the values of the lens parameters chosen, and for \( \theta_R = 26^\circ \), \(|z_1|/z_0\) has value 0.31 for \( R_T = a \), and 1.24 for \( R_T = 2a \). These values give the endpoints for the solid curves in Fig. 10. At \( z = 0 \), \(|V_L|\) is 6 dB lower than the value obtained from (29) due to the factor [1 - \((\frac{1}{2})\) erfc (s)]. In Fig. 10 a straight line, drawn dashed, has been used to connect \(|V_L|\) at the point \( z_1/z_0 \) and at the point \( z = 0 \), in order to indicate the general behavior of \(|V_L|\) in the range \( z_1 \leq z < 0 \).

The exponential variation of \(|V_L(z)|\) indicated in Fig. 10 has been observed by Smith and Wickramasinghe [27]. Working with a 12.5 MHz fused quartz lens \((R = 16 \text{ mm})\), they blocked its central portion so that none of the geometrically reflected rays reach the transducer for \(|z| > z_0 \). As a result, the voltage is due entirely to the leaky wave rays in this range of \( z \).

The curve of \(|V_G(z)|\) in Fig. 10 is the same as shown in Fig. 6 for \( z/z_0 < -1 \), except for a 2 dB reduction corresponding to \( R_T(0) \) of fused quartz. Deep minima in the AMS will occur when \(|V_L|\) and \(|V_G|\) are nearly equal. For \(|z| > z_0 \) the value of \(|V_L|\) will be greater than that of \(|V_G|\) implying that a method for selectively reducing \(|V_L|\) is required to achieve a series of deep minima. Attenuation in water, which has been neglected in drawing Fig. 10, is seen from (21) and (29) to slightly increase \(|V_L|\) over \(|V_G|\), while acoustic damping in the object has the reverse effect.

In expression (29), the amplitude term in the brackets is proportional to the attenuation \( \alpha \) of the Rayleigh wave resulting from reradiation into the water. The attenuation \( \alpha \) of the Rayleigh wave due to acoustic damping in the object does not appear in the amplitude term in the brackets, although it is in the exponential term. Dransfeld and Salzmann [19] give a simple approximate expression for \( \alpha \) that leads to the relation

\[
\frac{1}{2} \omega \approx \left( \rho_w/\rho_s \right) \left( \sin \theta_R \right)^2,
\]

where \( \rho_s \) and \( \rho_w \) are the mass densities of the object and water. Using this expression, the bracketed term in (29) becomes

\[
\frac{2 \omega \sqrt{\pi f^2 \sin \theta_R^{1/2}}}{(n \omega D)^{3/4}} \approx \frac{2 \pi^{1/4}}{F^{3/4}} \cdot \frac{\rho_w}{\rho_s} \cdot \left( \sin \theta_R^{1/2} \right) \left( f/R_T \right)^{3/2},
\]

which allows simple estimation of the amplitude of the leaky wave voltage relative to \( V_G \).

V. TOTAL VOLTAGE AND THE AMS

The total output voltage \( V(z) \) of the transducer for \( z < 0 \) is obtained by adding (21) and (29), taking into account their relative phases. For \( z > 0 \), \(|V_L|\) will decrease very rapidly to zero, and hence the total voltage will be nearly equal to \( V_G \).

As an example, we have plotted \(|V_G(z)|\), \(|V_L(z)|\), and \(|V(z)|\) in Fig. 11 for a 1.1 GHz sapphire lens looking at YIG. The dots in Fig. 11 represent measurements reported by Quate, Atalar, and Wickramasinghe [5]. The lens dimensions \( R = 105 \mu \text{m}, a = 75 \mu \text{m}, \) and \( D = 1230 \mu \text{m} \) give the parameters \( F = 1.1, z_0 = 0.82 \mu \text{m}, \) and \( f = 121 \mu \text{m} \). Attenuation in water was taken to be 0.204 dB/\mu m, and accounts for the fact that \(|V_L|\) increases to the left. The Rayleigh angle \( \theta_R = 24.6^\circ \) was chosen for the calculation to fit the measured data. This angle corresponds to a Rayleigh velocity \( V_R = 3600 \text{ m/s} \), which is in the range of reported values for YIG. The theoretical curve gives reasonable agreement with the measurements, except in the range \(-1 < z < 5 \) where diffraction effects in the lens rod are most important.

A. Acoustic Material Signature

The series of maxima and minima for \( z < 0 \) constitutes the AMS, and may be used to measure the Rayleigh velocity of the object. The minima occur when the phase difference between \( V_G \) and \( V_L \) is an odd multiple of \( \pi \). The phase difference is given by

\[
2k_w \left( 1 - \cos \theta_R \right) |z| + \left( \pi + |K - X| \right),
\]

provided that \( F > \frac{1}{2} \) so that \( \sin X \) does not change the sign for \( z < 0 \). When \( z \) is several times \( z_0 \), \( X = \pi/(2F) \) and the second term in (34) is independent of \( z \). In this case, the separation between minima is given by \((\frac{1}{2}) \lambda_w/(1 - \cos \theta_R)\), which was previously derived [14], [15]. For \( F < \frac{1}{2} \), additional \( \pi \) phase shift must be included in (34) as the zeros of \( \sin X \) are passed.

The accuracy with which the Rayleigh velocity can be determined depends on the number of minima that can be measured and their narrowness. Narrowness of the minima depends on the peak-to-valley ratio, which in turn depends on how close \( |V_G| \) is to \( |V_L| \). Thus, it is desirable to have \(|V_G|\) and \(|V_L|\) as closely equal to possible over a large range \( z < 0 \). In Fig. 11, crossing of the curves for \(|V_L|\) and \(|V_G|\) near a point where the phase difference (34) is \( 3\pi \) results in a very deep minimum. However, since \(|V_L|\) becomes greater than \(|V_G|\) to the left of this point, the next minimum is less deep.

To study the depth of the minima, consider the ratio of \(|V_G|\)
to $|V_L|$ in the range $|z| > z_0$. In this range $X = \pi/(2F)$ so that using (21), (24), (29) and taking $F \approx \lambda_w D/R_T^2$,

$$\frac{|V_G|}{|V_L|} = \left[ \frac{F^{3/4}}{\pi^{5/4}} \cdot \frac{f/R_T^{1/2}}{|K|} \sin \frac{\pi}{2F} \right] \frac{1}{2\alpha_k (\sin \theta_R)^{1/2}} \cdot \exp \left\{ 2z \right\} \left[ \alpha \tan \theta_R - \alpha_w \tan (\theta_R - 1) \right]. \quad (35)$$

The dependence of this ratio on $|z|$ is given by the last factor. For the typical case where $\alpha \tan \theta_R$ is large compared to $\alpha_w (\sec \theta_R - 1)$, the ratio is large for $|z|$ small and decreases to a minimum at

$$|z| = \left( \frac{1}{2} \right) |\alpha \tan \theta_R - \alpha_w (\sec \theta_R - 1)|. \quad (36)$$

If the minimum (36) is less than the distance from the edge of the lens to the focal plane, then $|z|$ may be increased beyond (36), in which case the ratio (35) will again increase.

Substituting the value of $|z|$ at the minimum into (35), and neglecting $\alpha_w (\sec \theta_R - 1)$ yields the minimum values of the ratio as

$$\frac{|V_G|}{|V_L|} = 0.65 \left[ \frac{F^{3/4}}{\pi^{5/4}} \cdot \frac{f}{R_T} \sin \frac{\pi}{2F} \right] \left[ \frac{1}{2\alpha_k} (\sin \theta_R)^{1/2} \right] \cdot \exp \left\{ 2z \right\} \left[ \alpha \tan \theta_R - \alpha_w \right]. \quad (37)$$

The last factor in (37) depends entirely on material parameters. Since $K$ is only weakly dependent on $\rho_R$, the middle factor depends primarily on the lens properties.

For the lens used to make the measurements shown in Fig. 11, and taking $|K| = 1$, the middle factor in (37) is 1.14. Since YIG has low acoustic damping $\alpha \approx \alpha_k$, and the last factor has value 0.65. Hence, the minimum value of $|V_G|/|V_L|$ is 0.48 which corresponds to a peak-to-valley ratio in the AMS of 9 dB. The AMS may be improved somewhat by the choice of $F$ and $R_T$.

### B. Effects of Damping and Anisotropy

Materials with intrinsic acoustic damping have $\alpha > \alpha_k$. For materials with small damping, having $\alpha > \alpha_k$ makes the ratio (37) closer to unity and thus improves the peak-to-valley ratio. However, materials with very large damping ($\alpha \gg \alpha_k$) will have the ratio (37) greater than unity. In this case, the peak-to-valley ratio improves as $\alpha$ decreases. This effect has been seen on steels with attenuation up to 6 dB per wavelength [30].

Anisotropy in the substrate has several effects that tend to reduce $|V_L|$. Beam steering of the Rayleigh wave in the surface will prevent surface waves from focusing on the lens axis. As a result, some of the returning rays in the lens rod will not travel in the plane containing the lens axis. Because of this additional tilt, these rays will give a lower contribution to the output voltage, or may even miss the transducer. Even without beam steering, the variation of $\delta R$ requires that the factor $\exp (-i2k_w z \cos \theta_R)$ in (29) be replaced by its average over azimuth angle in the transducer plane. The resulting phase cancellation reduces the magnitude of the average.

Another possible effect of anisotropy is typified by the (100) cut of Si. For this cut the Rayleigh wave exists only for limited ranges of azimuth angle. Outside this range it merges with a bulk shear wave [31]. Out of a total of 360° of azimuth angle, the Rayleigh wave exists only over about 224°. This fact effects $|V_L|$ by the factor 224/360, or 4 dB.

In Fig. 12, we have plotted $|V_G|^2$, $|V_L|^2$, normalized to $|V_A|^2$, for a 370 MHz sapphire lens looking at Si. The calculations include the 4 dB reduction in $|V_L|$ discussed above. It was assumed that $R = 380 \mu$m, $R_T = a = R \sin 55^\circ$ and $F = 0.8$ in order to compare with measurements made by Weglein [11] on Si using a lens of similar dimensions. The peak-to-valley ratio compares well with Weglein's measurements made on (100) Si. On the (111) cut of Si, on which Weglein also made measurements, the Rayleigh wave exists for all angles, but has a large variation of velocity with direction. Judging from the peak-to-valley ratio found for the (111) cut, beam steering and phase cancellation effects are more significant for this cut than the effect cited above is for the (100) cut.

### VI. Conclusion

Using a ray-optical approach, simple formulas have been derived that give the output voltage $V_G(z)$ due to the fields geometrically reflected at the object surface, and the voltage $V_L(z)$ due to the excitation and reradiation of the leaky wave. Contrary to previous assertions [5], ray optics does not lead to a simple sine $(kz)/z$ variation for $|V_G|$. Instead, $|V_G|$ shows the marked asymmetry observed for actual lenses. In particular, for $|z| > z_0$ and $F$ somewhat greater than $\frac{1}{2}$, $|V_G|$ varies as $1/|z|$. The expression for $V_L$ in the range $z < 0$ indicates that its magnitude decreases as a simple exponential, except for $z$ near zero.

The total voltage $V(z)$ predicted by the ray-optical formulas is in good agreement with measurements in the AMS region. Near focus, the formulas are not as accurate because effects such as diffraction in the lens rod have been neglected. Nevertheless, they do give a qualitative description of near focus behavior.

The formulas make it relatively easy to see how the various geometric and acoustic parameters of the lens and object influence $V(z)$. For example, it was shown how the parameters determine the peak-to-valley ratio in the AMS. To improve this ratio significantly, it is necessary to reduce $V_L$ as compared to $V_G$ by some modification of the microscope. One possible approach, suggested by the effects of anisotropy, is to break the cylindrical symmetry of the Rayleigh wave fields about the lens axis. This could be accomplished by notching the transducer on one side down to a radius slightly less than $\rho_R$.

### Appendix

#### Evaluation of Leaky Wave Fields

In order to evaluate the integral (10), we substitute for $P_1(k_x, z_1)$ its Fourier representation in terms of $\rho(x, z_1)$. With this substitution, and changing order of integration, one
obtains

$$p_L(x, z_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p_l(x', z_i) \left[ \int_{-\infty}^{x} G(0) e^{ikp(x-x')} \frac{k^2 - k_0^2}{k_x^2 - k_p^2} e^{ik_x(x-x')} \, dk_x \right] dx'. \quad (A1)$$

The integration over $k_x$ may be evaluated by deforming the path of integration, which initially lies along the real $k_x$ axis, into the upper half plane for $x - x' > 0$, or into the lower half plane for $x - x' < 0$. In the first case the pole at $k_x = k_p$ is intercepted, while in the second case the pole at $k_x = -k_p$ is intercepted. Taking the residues at the pole gives

$$p_L'(x, z_i) = -2\alpha_I e^{ikp x} \int_{-\infty}^{x} G(0) e^{ax'} \exp \left\{ -i \left[ k_w \sqrt{(x')^2 + (z_i)^2} + k_R x' \right] \right\} \frac{1}{k_0'} \frac{1}{((x')^2 + (z_i)^2)^{1/4}} \, dx'. \quad (A2)$$

The integrand of $(A4)$ has a stationary phase point $[25]$ when

$$\frac{d}{dx'} \left[ -k_w \sqrt{(x')^2 + (z_i)^2} - k_R x' \right] = 0. \quad (A5)$$

The solution of this equation for $x'$ is

$$x' = z_i \tan \theta_R = x_i, \quad (A6)$$

where $x_i$ is the point at which the incident ray making an angle $\theta_R$ with the $z$ axis intersects the surface of the solid. Expanding the phase to second order in $(x' - x_i)$ about the stationary point, and evaluating all amplitude terms at the stationary point, the integral in $(A4)$ may be approximately by the method of stationary phase $[25]$ as

$$p_L'(x, z_i) = -2\alpha_I e^{ikp(x-x_i)} p_l(x_i, z_i) \int_{-\infty}^{x} \exp \left\{ -i \frac{k_w}{2z_i} (\cos \theta_R)^3 (x' - x_i)^2 \right\} \, dx'. \quad (A7)$$
Making a change of variable in the integration, the integral can be expressed in terms of the complementary error function [32]. Thus,

\[
p_l^2(x_i, z_i) = -\frac{2a_1}{k_w} \sqrt{\frac{2\pi |z_i|}{\cos \theta R^3}} e^{-i\alpha z_i} e^{ik_w(x-x_i)}
\]

\[
p_l(x_i, z_i) \cdot [1 - (\frac{1}{2}) \text{erfc}(s)],
\]

where

\[
s = (x - x_i) \sqrt{\frac{k_w (\cos \theta R^3)^2}{2 |z_i|}} e^{i\alpha z_i}.
\]

For \((x - x_i)\) large and positive, \(s\) has large magnitude and lies in the first quadrant so that \(|\text{erfc}(s)| << 1\). In this limit, (A8) reduces to (12). Numerically, \(|\text{erfc}(s)| < 0.1\) for \(|s| > \sqrt{\pi}\), which occurs when

\[
x - x_i > [\frac{1}{2} |z_i| \lambda_w (\cos \theta R^3)]^{1/2}
\]

(A10)

If \(|z_i| >> \lambda_w\), then \(\text{erfc}(s)\) may be neglected a short distance to the right of the point \(x_i\).

In evaluating the leaky wave contribution to the output voltage we are primarily concerned with the field radiated about the point \(-x_i\), which is located symmetrically about the \(z\) axis from the launch point \(x_i\). Using this value of \(x\) in (A10), together with (A6), it is seen that \(\text{erfc}(s)\) can be neglected at \(-x_i\) provided that

\[
|z_i| > \frac{\lambda_w}{4 |\cos \theta R^3 |} (\sin \theta R^3 )^2
\]

which is on the order of \(\lambda_w\).

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